Polyhedral Results for A Class of Cardinality Constrained Submodular Minimization Problems

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- c: Cost vector (of size n)
- b: Budget
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Mean risk combinatorial optimization
[Shen et al '03, Atamturk'08, Nikolova'10, Baumann et al'13]

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Polyhedral description/relaxation of $\mathrm{conv}(\mathcal{P} \cap X)$ where

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Simple case: Cardinality constraint $X = \{x \in [0,1]^n:\ e^\top x \leq K\}$

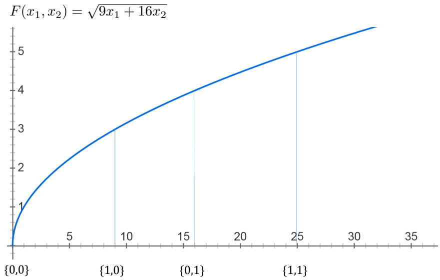
Submodular Functions

Definition

A function $F : \{0,1\}^n \to \mathbb{R}$ is *submodular* if

$$F(x + e^i) - F(x) \geq F(y + e^i) - F(y) \;\; \forall\, x \leq y, \text{ s.t. } x_i = y_i = 0$$

If f is concave and $a \in \mathbb{R}^n_+$ then $F(x) := f(a^\top x)$ is submodular.



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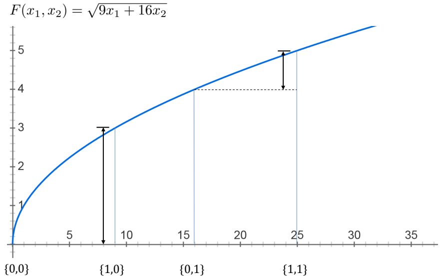

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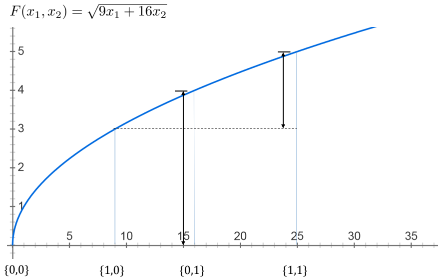

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- Submodular inequalities in MINLP:
  Atamturk et al. '08,'09,'12, Tawarmalani'10, A.+Papageorgiou'13, Bauman et al.'13 ....

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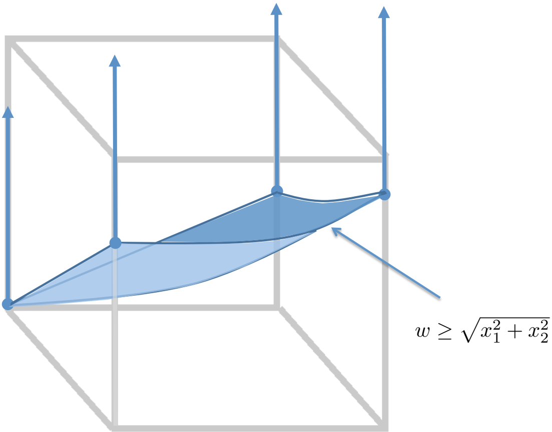
- Goal: Extend the submodular inequalities to incorporate cardinality constraint

Illustration



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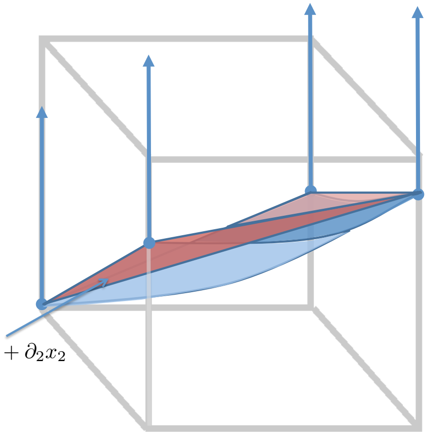

$w \geq \partial_0 + \partial_1 x_1 + \partial_2 x_2$

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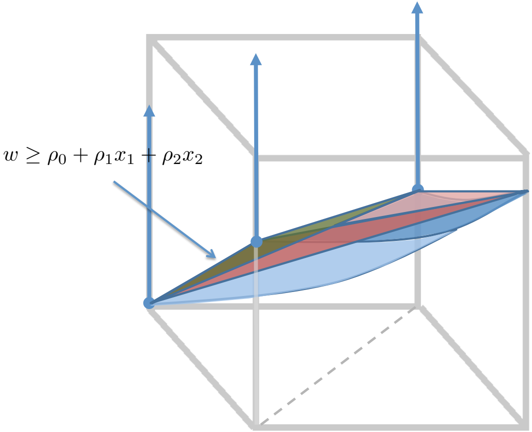

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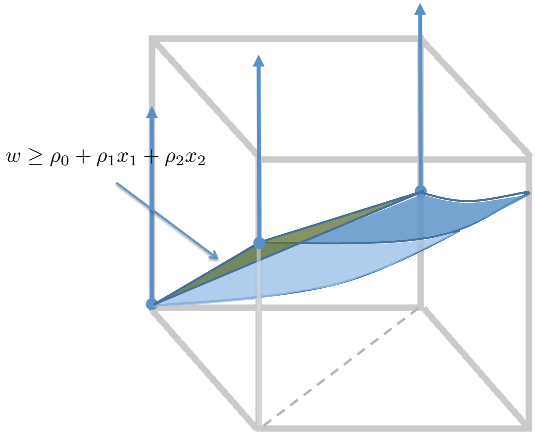

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- Computational results for mean-risk knapsack

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- The primal LP:

$$\begin{aligned} \min \quad & \sum_{S \in \mathcal{S}} f(|S|) P(S) \\ \text{s.t.} \quad & \sum_{S \in \mathcal{S} : i \in S} P(S) = x_i \qquad \forall i \in [n] \\ & \sum_{S \in \mathcal{S}} P(S) = 1, \; P(S) \geq 0 \quad \forall\, S \in \mathcal{S} \end{aligned}$$

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$$\rho_i = \begin{cases} f(0) & i = 0 \\ f(i) - f(i-1) & 1 \leq i \leq i_0 \\ \frac{f(K)-f(i_0)}{K-i_0} & i_0 < i \leq n \end{cases}$$

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- Note that $\rho_i = \partial_i$ for $i \leq i_0$

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- The lhs of $(\star)$:

$$\partial_0 + \sum_{i \in S_1} \partial_i + \sum_{i \in S_2} \rho_i \leq f(i_1) + i_2 \cdot \frac{f(K) - f(i_0)}{K - i_0}$$

by construction and validity of the usual submodular inequality corresponding to S_1

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- The above inequality can be shown using concavity of $f(i)$ and $i_1 \leq i_0 \leq K$

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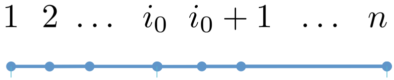

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$$1 \quad 2 \quad \ldots \quad i_0 \quad i_0+1 \quad \ldots \quad n$$

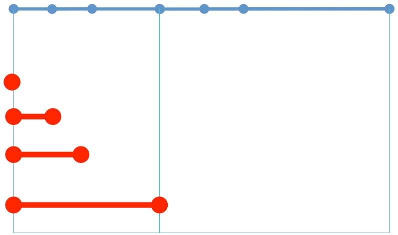
$\emptyset$

$\{1\}$

$\{1,2\}$

$\vdots$

$\{1,2,\ldots,i_0\}$



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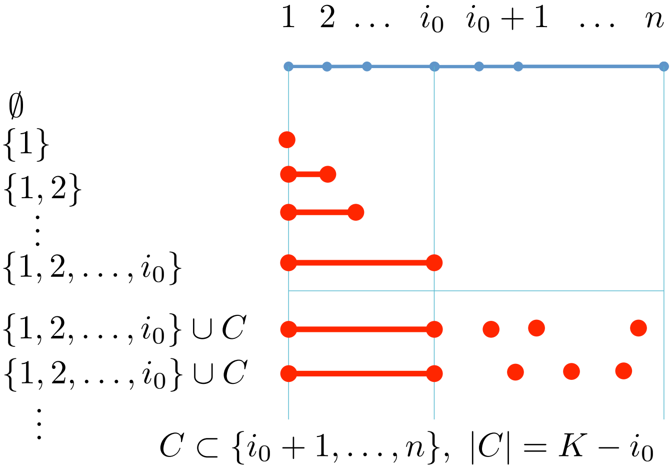

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  $\sum_{C: C \in \mathcal{C}} Q(C) = x_i, \quad i = i_0 + 1, \ldots, n$ where
  $\mathcal{C} = \{C \subset \{i_0 + 1, \ldots, n\} : \ |C| = K - i_0\}$

- $P(\{1, \ldots, i_0\} \cup C) = Q(C)$ for all $C \in \mathcal{C}$

- The constructed primal solution is feasible and has the same objective value as that of the dual problem

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- Resulting (facet-defining) lifted inequality (LI):

$$\partial_0 + \sum_{i \in S} \partial_i x_i + \sum_{i \notin S} \lambda_i x_i \leq w$$

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  is valid for $\operatorname{conv}(\mathcal{P} \cap X)$
- Proof: $0 \leq \gamma_i \leq \lambda_i$ for all $i > K$

Weighted Case: Remarks on Approximate Lifting

- The approximate lifting inequality is a version of the unweighted inequality corresponding to $i_0 = K - 1$
- $\partial_i \leq \gamma_i$ for all $i > K$ with the inequality being strict when f is strictly monotone (e.g. square root)
- Can be computed in $O(n \log n)$ time

Computations: Instances

- Mean-risk Knapsack:

$$\min \left\{ -\sum_i \mu_i x_i + \lambda \sqrt{\sum_i \sigma_i^2 x_i} : \; c^\top x \le b, \; x \in \{0,1\}^n \right\}$$

- 1,080 unweighted and 1,080 weighted instances randomly generated ($n = 50, 80, 100$)

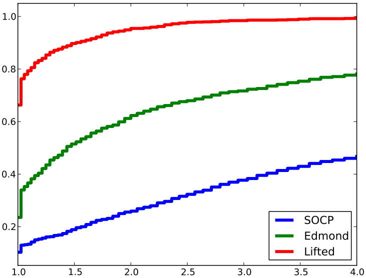
- Knapsack to Cardinality: Let $c_1 \le \cdots \le c_n$ and pick K such that $c_1 + \cdots + c_K \le b$ and $c_1 + \cdots + c_{K+1} > b$

Computations: Experiments

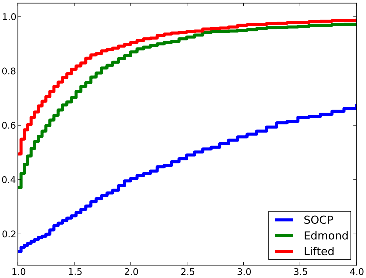
- Compare branch-and-cut time and nodes for
  - Mixed Integer SOCP formulation (Replace x_i by x_i^2) [SCOP]
  - SOCP + 5 rounds of usual submodular inequalities [Edmonds]
  - SOCP + 5 rounds of exact separated inequalities in the unweighted case, or SOCP + 5 rounds of approximate lifted inequality in the weighted case [Lifted]

- Implemented in Python + Gurobi 5.6

Computations: Performance profiles (Time)



Unweighted



Weighted

Computations: Averages

	SOCP	Edmonds	Lifted
Unweighted			
Time:	703	528	295
Nodes:	221,111	135,444	75,811
Weighted			
Time:	208	132	122
Nodes:	59,779	34,366	27,928

Concluding Remarks

$$\mathcal{P} = \left\{(w,x) \in \mathbb{R} \times \{0,1\}^n : \; w \geq f(a^\top x)\right\} \quad X = \{x \in [0,1]^n : \; e^\top x \leq K\}$$

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- Summary:
  - Unweighted case (identical a_i's): Complete description of conv($\mathcal{P} \cap X$)
  - Weighted case (general a_i): Family of facets/valid inequalities by lifting
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- Some open issues:
  - Handle correlations $\sqrt{x^\top \Sigma x}$ by introducing new variables for cross terms
  - Mixed integer setting
  - "Submodular relaxations" of other classes of MINLP