Towards Efficient Higher-Order Semidefinite Relaxations for Max-Cut

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The Max-Cut Problem

Given a graph $G = (V, E)$ with $|V| = n$ and weights w_{ij} for all edges $(i, j) \in E$, find an edge-cut of maximum weight, i.e. find a set $S \subseteq V$ s.t. the sum of the weights of the edges with one end in S and the other in $V \setminus S$ is maximum.

- We assume wlog that $w_{ii} = 0$ for all $i \in V$, and that G is complete (assign $w_{ij} = 0$ if edge $ij \notin E$).
- Let $x \in \{-1, +1\}^n$ represent any cut in the graph then max-cut may be formulated as:

$$
\begin{array}{rll}
z_{mc} := & \max & \sum_{i=1}^{n} \sum_{j=i+1}^{n} w_{ij} \left(\frac{1 - x_i x_j}{2} \right) = x^T Q x \\
& \text{s.t.} & x_i^2 = 1, i = 1, \ldots, n,
\end{array}
$$

where $Q = \frac{1}{4} (\text{Diag}(We) - W)$.

The Basic Semidefinite Relaxation of Max-Cut

Consider the change of variable $X = xx^T, x \in \{\pm 1\}^n$.
Then $X_{ij} = x_i x_j$ and since $x^T Q x = \langle Q, xx^T \rangle$, max-cut is equivalent to

$$\begin{array}{ll} \max & \langle Q, X \rangle \\ \text{s.t.} & \operatorname{diag}(X) = e \\ & \operatorname{rank}(X) = 1 \\ & X \succeq 0. \end{array}$$

Removing the rank constraint, we obtain the basic semidefinite relaxation of max-cut:

$$z_{sdp} := \begin{array}{ll} \max & \langle Q, X \rangle \\ \text{s.t.} & \operatorname{diag}(X) = e \\ & X \succeq 0. \end{array}$$

The Cut Polytope and the Correlation Matrices

The convex hull of the 2^{n-1} feasible solutions for max-cut is called the cut polytope:

$$CUT_n := \text{conv}\{xx^T : x \in \{-1, 1\}^n\}.$$

Thus Max-Cut can be formulated as

$$z_{mc} = \max\{\langle Q, X\rangle : \; X \in CUT_n\}.$$

CUT_n is contained in the set of correlation matrices:

$$C_n := \{X : \text{diag}(X) = e, \; X \succeq 0\}$$

and therefore

$$z_{mc} = \max\{\langle Q, X\rangle : X \in CUT_n\} \leq \max\{\langle Q, X\rangle : X \in C_n\} = z_{sdp}$$

The Metric Polytope

Deza, Laurent (1997): Hypermetric Inequalities

Consider $x \in \{-1, 1\}^n$, $f = (1, 1, 1, 0, \ldots, 0)^T \Rightarrow |f^T x| \geq 1$.

- Results in $x^T f\ f^T x = \langle ff^T, xx^T \rangle = \langle ff^T, X \rangle \geq 1$.
- Can be applied to any triangle $i < j < k$.
- Nonzeros of f can also be -1.

There are $4\binom{n}{3}$ such triangle inequality constraints.
We collect them in the metric polytope

$$MET_n := \{X : f^T X f \geq 1 \text{ where } f \text{ has 3 nonzeros} \in \{-1, 1\}.\}$$

- Barahona, Jünger, Reinelt (1989): computational experiments, LP relaxation very efficient for sparse graphs.
- Pardella, Liers (2008): computations with 2d spinglass problems of sizes larger than 1000×1000.
- Weak results once density of graph grows.

Other Classes of Inequalities

- If $f \in \{-1, 0, 1\}$ with $f^T f = t$, and t odd, we get odd-clique inequalities:

$$\{X : f^T X f \geq 1 \text{ where } f \text{ has } t \text{ nonzeros} \in \{-1, 1\}\}.$$

- Many other classes of facets of CUT_n are known but they are often difficult to separate, and no substantial computational experiments available.

Higher-Order Relaxations

There are several hierarchies of relaxations for 0-1 optimization problems, including:

- Sherali-Adams RLT procedure
- Lovász-Schrijver liftings
- Higher liftings by A. and Wolkowicz (2002) and Lasserre (2002); also sums-of-squares relaxations by Parrilo (2000).

They attain the integer optimum in n lifting steps, but at each step the dimension of the problem grows.

Even the first nontrivial lifting step in the SDP hierarchies leads to SDP problems that are computationally out of reach even for, say, $n \approx 50$.

Now: Improved relaxations for which the matrix dimension remains n.

Key Observation

We can take any subset $I \subseteq \{1, 2, \ldots, n\}$ with $|I| = k$ and consider X_I, the principal submatrix of X indexed by I.

Key Observation

If $X \in CUT_n$ then $X_I \in CUT_k$.

This can be expressed as

$$X_I = \sum_j \lambda_j \bar{v}_j \bar{v}_j^T, \quad \lambda \geq 0, \quad \sum_j \lambda_j = 1,$$

where $\bar{v}_j \in \{\pm 1\}^k$ runs through the 2^{k-1} cuts in CUT_k.

A New "Hierarchy" of Relaxations

This leads to a new sequence of relaxations for max-cut indexed by k:

$$\begin{array}{rll} z_{sdp-met-k} = & \max & \langle Q, X \rangle \\ & \text{s.t.} & \operatorname{diag}(X) = e \\ & & X \succeq 0 \\ & & \text{triangle inequalities on } X \\ & & X_I \in CUT_k \text{ for all } I \text{ with } |I| = k. \end{array}$$

- As k approaches n, we get better and better bounds, and if $k = n$ we get the exact solution.
- For $k \leq 4$ we have $z_{sdp-met} = z_{sdp-met-k}$ because $MET_k = CUT_k$ for $k \leq 4$.
- Smallest interesting case: $k = 5$.

Illustrative Example (Laurent (2004))

$$Q = \frac{1}{2}\begin{pmatrix} 0 & 1 & 1 & 1 & -2 & -1 & 0 \\ 1 & 0 & 1 & 1 & -2 & 0 & -1 \\ 1 & 1 & 0 & 1 & -2 & -1 & 0 \\ 1 & 1 & 1 & 0 & -2 & 0 & -1 \\ -2 & -2 & -2 & -2 & 0 & 1 & 1 \\ -1 & 0 & -1 & 0 & 1 & 0 & -1 \\ 0 & -1 & 0 & -1 & 1 & -1 & 0 \end{pmatrix}$$

Relaxation	Bound
C_7	6.9518
$C_7 \cap MET_7$	6.0584
$C_7 \cap MET_7$ plus best CUT_5: $\{1, 3, 5, 6, 7\}$	5.9800
$C_7 \cap MET_7$ plus best CUT_6: $\{1, 2, 3, 4, 5, 6\}$	5.9412
$C_7 \cap MET_7$ plus all CUT_5s	5.8000
A.-Wolkowicz	5.7075
$C_7 \cap MET_7$ plus all CUT_6s	5.6667
Lasserre level-2	5.6152
$C_7 \cap MET_7$ plus CUT_7	5.0000

Related Earlier Work

Previous work using this idea in connection with polyhedral relaxations:

- Using small-dimensional polytopes to improve relaxations is a well-known idea, see e.g. Applegate et al. (2001).
- Also similar to the recent work on target cuts in Buchheim, Liers and Oswald (2008), and the lifting and separation of Bonato et al. (2011).
- In most earlier work, an outer description of the small polytope is used to lift local cuts to cuts for the original problem.
- We believe that an inner description for the small polytope has algorithmic advantages.

Additional Observations

- This approach works for graph optimization problems with the property that restriction to node-induced subgraphs results in a similar optimization problem of smaller dimension.
  Other candidate problems include max-stable-set /max-clique and graph coloring.
- For each I we add 2^{k-1} nonnegative variables and $\binom{k}{2}$ new equations.
- Adding the constraints for all I at once is computationally inefficient, so the challenge is to identify good choices of I.

Selecting the Best Subset I

Given $X \in C_n \cap MET_n$, we want to identify a subset I with $|I| = k$ such that $X_I \notin CUT_k$.

The problem of finding I of cardinality k and maximizing the distance of the corresponding polytope is:

$$
\begin{array}{rl}
\max & d \\
\text{s.t.} & Be_n = e_k \\
& Be_k \leq e_n \\
& B \in \{0, 1\}^{n \times k} \\
& d = \left\{ \min_{e^T\lambda=1,\, \lambda\geq 0} || \operatorname{triu}\left(B^T X B\right) - Q\lambda || \right\}
\end{array}
$$

With some manipulations, this problem can be expressed as a 0-1 SOCO problem.

Computational Setup

- The relaxation $C \cap MET$ is usually quite accurate on smaller instances with n up to $n \approx 50$, so we consider instances with $60 \leq n \leq 100$.
- We include in each round the best 50 new subsets I with $|I| = 5$.
- The resulting SDP is solved using an interior-point code (SDPT3).
- Triangles are separated by complete enumeration.

Computational Setup (ctd)

We focus on the case $k = 5$.

Start:
- Find optimal solution $X \in C_n \cap MET_n$

Iteration:
(a) Determine subsets I_r with $|I_r| = 5$
(b) Resolve with $X_{I_r} \in CUT_5$ yielding new X
(c) Add triangle inequalities violated by X
(d) Purge inactive triangles
(e) Resolve with new triangles added yielding new X

Note: after (e) the condition $X \in C_n \cap MET_n$ is not guaranteed to hold. It could be enforced by repeating (c),(d) and (e) until all triangles inequalities are satisfied again.

Focus on One Instance

We select $n = 70$ and adjacency matrix with density of 50%, edge weights are integers between -10 and 10.
At start we get:

$$z_C = 996.1 \quad z_{C \cap MET} = 872.3, \quad z_{mc} = 856$$

round	bound	min s_l	# sets l	# triangles
1	868.2	0.41	48	670
2	865.9	0.55	94	602
3	864.1	0.54	138	516
4	862.4	0.56	183	509
.	.	.	.	.
10	858.3	0.76	344	416

Preliminary Computational Results

Random graphs, density 50 %, edge weights between -10 and 10

n	C	$C \cap M$	new	cut	% gap left
70	996.2	872.3	858.3	856	0.14
80	1317.2	1181.6	1162.6	1152	0.36
90	1491.1	1335.6	1307.8	1297	0.28
100	1959.6	1772.2	1745.8	1698	0.64

Random dense graphs, edge weights between 1 and 10

n	C	$C \cap M$	new	cut	% gap left
70	6807.1	6725.9	6712.9	6693	0.60
80	8741.6	8639.6	8623.2	8604	0.54
90	11217.8	11109.4	11092.6	11070	0.57
100	13718.9	13593.3	13575.1	13530	0.71

Current and Future Work

- Improve the separation
- Experiment with subsets of larger sizes
- Solve the resulting SDP problems more efficiently
- Apply to other problem classes
- Incorporate into Branch-and-Bound (BiqMac)

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Thank you for your attention.