A feasibility pump for nonconvex MINLP

MINLP workshop @CMU, 4 June 2014

Pietro Belotti
Xpress Optimization Team
UK

Timo Berthold
Xpress Optimization Team
Germany

MINLP

$$\begin{array}{rl} \min & f(\boldsymbol{x}) \\ \text{s.t.} & \boldsymbol{g}(\boldsymbol{x}) \leq \boldsymbol{0} \\ & \boldsymbol{x} \in \mathbb{Z}^p \times \mathbb{R}^{n-p} \cap [\boldsymbol{\ell}, \boldsymbol{u}] \end{array}$$

Assume f and $\boldsymbol{g} : \mathbb{R}^n \to \mathbb{R}^m$ are factorable, i.e. are composed by combining variables and constants with operators from a finite set $\mathcal{O} = \{+, -, \times, \hat{}, /, \log, \sin, \cos\}$

Feasibility pump heuristics[1]

Feasibility Pump heuristics are suited for problems where

- integrality constraints: $\boldsymbol{x} \in \mathcal{I} = \mathbb{Z}^p \times \mathbb{R}^{n-p}$ and
- continuous constraints: $\boldsymbol{x} \in \mathcal{C} = \{\boldsymbol{x} \in \mathbb{R}^n : \boldsymbol{g}(\boldsymbol{x}) \leq \mathbf{0}\}$

coexist, such as MILP and MINLP.

Idea: generate two sequences of (possibly infeasible) solutions:

- $(\hat{\boldsymbol{x}}^k)_{k \in N}$: satisfies integrality, i.e., $\hat{\boldsymbol{x}}^k \in \mathcal{I}$
- $(\tilde{\boldsymbol{x}}^k)_{k \in N}$: satisfies continuous constraints, i.e., $\tilde{\boldsymbol{x}}^k \in \mathcal{C}$

The next point of sequence $(\tilde{\boldsymbol{x}}^k)$ is the closest to the previous point of sequence $(\hat{\boldsymbol{x}}^k)$ and viceversa.

[1] M. Fischetti, F. Glover, A. Lodi, "The feasibility pump", *Math. Prog.* 104(1):91-104, 2005.

Feasibility pump heuristics

Initialization: $\tilde{\boldsymbol{x}}^0 \in \mathcal{C}$.

1. $k \leftarrow 1$
2. repeat
3. $\quad \hat{\boldsymbol{x}}^k \in \text{argmin}_{\boldsymbol{x} \in \mathcal{I}} ||\boldsymbol{x} - \tilde{\boldsymbol{x}}^{k-1}||$
4. $\quad \tilde{\boldsymbol{x}}^k \in \text{argmin}_{\boldsymbol{x} \in \mathcal{C}} ||\boldsymbol{x} - \hat{\boldsymbol{x}}^k||$
5. $\quad k \leftarrow k + 1$
6. until $\hat{\boldsymbol{x}}^k$ or $\tilde{\boldsymbol{x}}^k$ is feasible

The MILP feasibility pump

Consider

$$(\textbf{MILP}) \quad \min\{\boldsymbol{c}^\top \boldsymbol{x} : A\boldsymbol{x} \leq \boldsymbol{b}, \boldsymbol{x} \in \mathbb{Z}^p \times \mathbb{R}^{n-p}\}$$

Set $\tilde{\boldsymbol{x}}^0 \in \operatorname{argmin}\{\boldsymbol{c}^\top \boldsymbol{x} : A\boldsymbol{x} \leq \boldsymbol{b}\}$ and $k = 1$. Then

$$\begin{aligned} \hat{\boldsymbol{x}}^k &= \lfloor \tilde{\boldsymbol{x}}^k \rceil \\ \tilde{\boldsymbol{x}}^k &\in \operatorname{argmin}\{\|\boldsymbol{x} - \hat{\boldsymbol{x}}^{k-1}\|_1 : A\boldsymbol{x} \leq \boldsymbol{b}\} \end{aligned}$$

Variants

The original FP finds a feasible, but often bad, initial solution.

- The objective function $\boldsymbol{c}^\top \boldsymbol{x}$ is ignored

$\Rightarrow$ Objective FP[2]:

$$\begin{aligned} \hat{\boldsymbol{x}}^k &= \lfloor \tilde{\boldsymbol{x}}^k \rceil \\ \tilde{\boldsymbol{x}}^k &\in \operatorname{argmin}\{\alpha \|\boldsymbol{x} - \hat{\boldsymbol{x}}^{k-1}\|_1 + \beta \boldsymbol{c}^\top \boldsymbol{x} : A\boldsymbol{x} \le \boldsymbol{b}\} \end{aligned}$$

Feasibility Pump 2.0[3]:

- better rounding heuristics than $\lfloor \boldsymbol{x} \rceil$
- perturbation in the event of cycling

[2] T. Achterberg, T. Berthold, "Improving the feasibility pump". *Discrete Optimization* 4(1):77-86, 2007.

[3] M. Fischetti, D. Salvagnin, "Feasibility pump 2.0", *Math. Prog. Comp.* 1(2-3):201-222, 2009.

FP for convex MINLP[4]

The first attempt to port the FP to the MINLP realm.

- The two sequences obtained by solving two simpler problems:
  - an NLP problem
  - a MILP approximation of the MINLP
- MILP provided by Outer Approximation:

$$\begin{aligned} \min \quad & z \\ \text{s.t.} \quad & f(\tilde{\boldsymbol{x}}^k) + \nabla f(\tilde{\boldsymbol{x}}^k)(\boldsymbol{x} - \tilde{\boldsymbol{x}}^k) \le z \\ & g_i(\tilde{\boldsymbol{x}}^k) + \nabla g_i(\tilde{\boldsymbol{x}}^k)(\boldsymbol{x} - \tilde{\boldsymbol{x}}^k) \le 0 \quad \forall i = 1, 2, \ldots, m \end{aligned}$$

[4] P. Bonami, G. Cornuéjols, A. Lodi, F. Margot, “A Feasibility Pump for Mixed Integer Nonlinear Programs.” *Math. Prog.* 119(2):331-352, 2009.

Nonconvex MINLP

- integrality constraints: $\boldsymbol{x} \in \mathcal{I} = \mathbb{Z}^p \times \mathbb{R}^{n-p}$ and
- nonconvex continuous constraints:

$$\boldsymbol{x} \in \mathcal{C} = \{\boldsymbol{x} \in \mathbb{R}^n : \boldsymbol{g}(\boldsymbol{x}) \leq \boldsymbol{0}\}$$

Previous attempts try to solve (even heuristically) the continuous nonconvex problem through a global solver or multi-start[5]

[5] C. D'Ambrosio, A. Frangioni, L. Liberti, A. Lodi, "Experiments with a feasibility pump approach for nonconvex MINLPs", Proceedings of SEA 2010.

Nonconvex MINLP: idea

Take advantage of features of both the solver and the problem.

- convexification cuts: use a valid LP relaxation rather than Outer Approximation
- nonlinearity: use second-order information
- MILP: diversify search

FP: component problems

Global optimization techniques construct an (MI)LP relaxation by adding q auxiliary variables.

$$
\begin{array}{lll}
(\textbf{MILP}) & \min & x_{n+q} \\
& \text{s.t.} & A\boldsymbol{x} \geq \boldsymbol{b} \\
& & \boldsymbol{x} \in (\mathbb{Z}^s \times \mathbb{R}^{n+q-s}) \cap [\boldsymbol{\ell}', \boldsymbol{u}']
\end{array}
$$

so redefine our components:

$$
\begin{array}{rcl}
\mathcal{C} & = & \{\boldsymbol{x} \in \mathbb{R}^n \cap [\boldsymbol{\ell}, \boldsymbol{u}] : \boldsymbol{g}(\boldsymbol{x}) \leq \boldsymbol{0}\} \\
\mathcal{I} & = & \{\boldsymbol{x} \in \mathbb{Z}^s \times \mathbb{R}^{n+q-s} \cap [\boldsymbol{\ell}', \boldsymbol{u}'] : A\boldsymbol{x} \geq \boldsymbol{b}\}
\end{array}
$$

FP for nonconvex MINLPs

Initialization: $\tilde{\boldsymbol{x}}^0 \in \operatorname{argmin}\{f(\boldsymbol{x}) : \boldsymbol{g}(\boldsymbol{x}) \leq 0\} \in \mathcal{C}$.

1. $k \leftarrow 1$
2. repeat
3. $\quad \hat{\boldsymbol{x}}^k \in \operatorname{argmin}_{\boldsymbol{x} \in \mathcal{I}} \Delta'(\boldsymbol{x}, \tilde{\boldsymbol{x}}^{k-1})$
4. $\quad \tilde{\boldsymbol{x}}^k \in \operatorname{argmin}_{\boldsymbol{x} \in \mathcal{C}} \Delta''(\boldsymbol{x}, \hat{\boldsymbol{x}}^k)$
5. $\quad k \leftarrow k + 1$
6. until $\hat{\boldsymbol{x}}^k$ or $\tilde{\boldsymbol{x}}^k$ is feasible

Δ' and Δ'' combine lp distance, objective function, and second order information from the problem

Second order information

Consider the (locally!) optimal solution $\tilde{\boldsymbol{x}}^0$ of the continuous relaxation of the MINLP.

- If no active constraints, the Hessian of the objective is PSD: $\nabla^2 f(\tilde{\boldsymbol{x}}^0) \succeq 0$
- The Hessian of the Lagrangian provides useful info

Using second order information and objective function

Construct PSD matrix P from the Hessian of the Lagrangian by

- Obtaining a projection on the cone of PSD matrices

... (by eliminating negative eigenvalues)

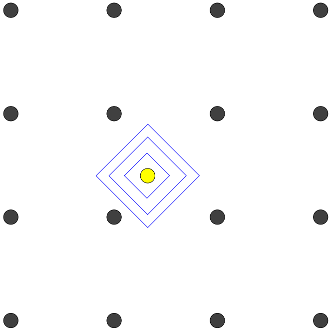
Then use:

MILP: $\Delta'(\boldsymbol{x}, \bar{\boldsymbol{x}}) := \alpha' ||\boldsymbol{x} - \bar{\boldsymbol{x}}||_1 + \beta' ||P^{\frac{1}{2}}(\boldsymbol{x} - \bar{\boldsymbol{x}})||_1 + \gamma' \boldsymbol{c}^\top \boldsymbol{x}$

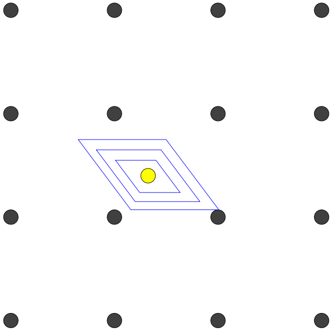
NLP: $\Delta''(\boldsymbol{x}, \bar{\boldsymbol{x}}) := \alpha'' ||\boldsymbol{x} - \bar{\boldsymbol{x}}||_2 + \beta'' (\boldsymbol{x} - \bar{\boldsymbol{x}})^\top P (\boldsymbol{x} - \bar{\boldsymbol{x}}) + \gamma'' f(\boldsymbol{x})$

FICO

Second-order information: intuition

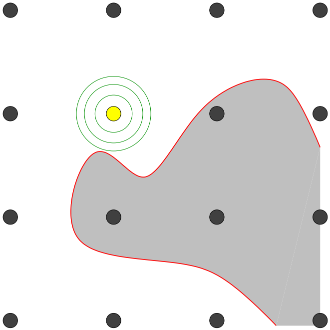


Second-order information: intuition



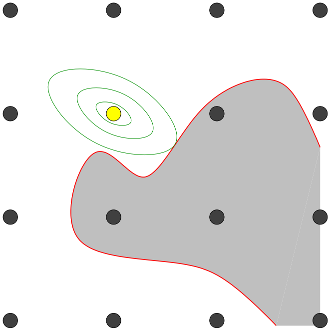
FICO

Second-order information: the NLP case

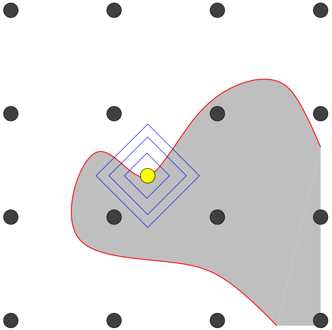


FICO

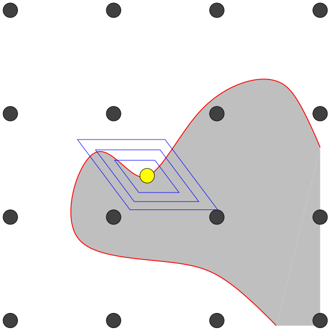
Second-order information: the NLP case



Second-order information: used in the MILP too



Second-order information: used in the MILP too



More features

A hierarchy of MILP solvers, from exact&slow to lousy&fast:

1. Solve MILP with node, time limit, emphasis on feasibility
2. Solve MILP with stricter node, time limit
3. RENS on MILP
4. SCIP's own Feasibility Pump
5. Couenne's rounding-based procedure
6. Round solution to closest integer

- If succeeded five times in a row, switch to cheaper method
- If current method returns no solution, switch to more expensive one and rerun

Implementation details

Implemented in Couenne, an Open-Source MINLP solver[6]

- Available in `stable/0.4` (but tested with `trunk`)
- MILP + heuristics: SCIP[7]
- NLP solver: Ipopt[8]

Tested on 218 MINLP instances from MINLPLIB[9]

---

[6] `http://www.coin-or.org/Couenne`
[7] `http://scip.zib.de`
[8] `http://www.coin-or.org/Ipopt`
[9] `http://www.gamsworld.org/minlp/minlplib.htm`

FICO

Diving right into the storm[10]

Andrea/Antonio/Claudia/Leo's FP found a solution for around 200 out of 243 test instances (compare with 157/218)

Of the stormtroopers' results for the 65 "hard" instances (instances for which only one solver succeeded):

- 15 times, we find a better solution;
- 21 times, it is worse

[10] D'Ambrosio et al., "A storm of feasibility pumps for nonconvex MINLP." *Math. Prog.* 136:2 (2012), 375-402.

FICO

Performance of different versions (single call)

default: ℓ_1 distance for MILP, solved exactly, no cuts

cuts: separate infeasible solutions through conv. cuts

hierarchy: Dynamically chooses MILP methods

hessian: Increase weight for Hessian, decrease for ℓ_2 dist.

objective: Same as hessian plus decreasing for objective

simple: Rounding to nearest integer instead of MILP

setting	feasible	better : worse	time (sgm)
default	150	–	14.9
cuts	155	24 : 48	13.6
hierarchy	157	23 : 20	14.0
hessian	154	25 : 16	22.8
objective	138	45 : 30	23.9
simple	97	17 : 78	12.1

FICO

MINLP capabilities in the Xpress Optimizer

Heuristics like this can be beneficial for commercial solvers.

In particular for the Xpress Optimizer:

- MIQCQP: can solve efficiently large problems
- MISOCP: Fastest solver[11], with effective bound reduction and presolver.
- Both would benefit from a sophisticated heuristic that takes quadratic elements and second-order info into account
- (MI)NLP: local solver (both first- and second-order NLP); rounding heuristics for integer variables to find good local MINLP solutions.

[11] Check http://plato.asu.edu/ftp/socp.html

Thank You

Pietro Belotti
pietrobelotti@fico.com