# Recent results on solving QCQPs, and related topics

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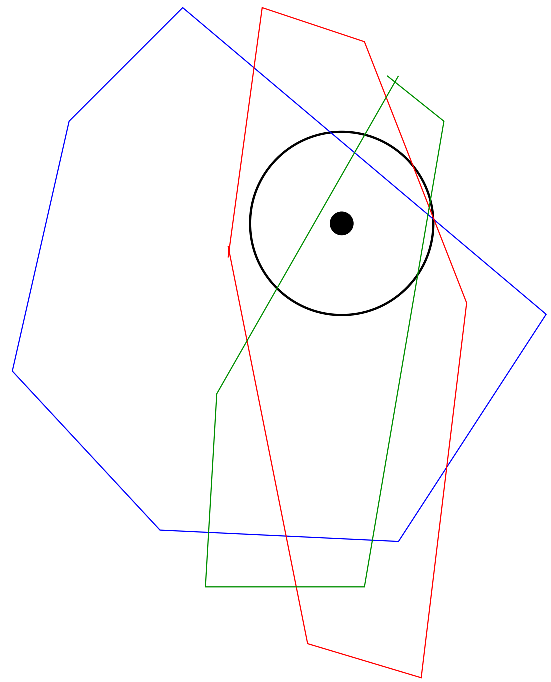
## Three problems

**1.** The "SUV" problem

- given full-dimensional polyhedra $P^1, \ldots, P^K$ in $\mathbb{R}^d$,
- find a point closest to the origin *not* contained inside any of the $P^h$.

$$\begin{aligned} \min \ & \|x\|^2 \\ s.t. \ & x \in \mathbb{R}^d - \bigcup_{h=1}^{K} \mathrm{int}(P^h), \end{aligned}$$

(application: X-ray lythography)



- Typical values for $d$ (dimension): less than 20; usually even smaller
- Typical values for $K$ (number of polyhedra): possibly hundreds, but often less than 50
- Very hard problem

**2.** Cardinality constrained, convex quadratic programming.

$$\min \; x^T Q x + c^T x$$
$$\text{s.t.} \quad Ax \leq b$$
$$x \geq 0, \quad \|x\|_0 \leq k$$

$\|x\|_0$ = number of nonzero entries in $x$.

- $Q \succeq 0$
- $x \in \mathbb{R}^n$ for $n$ possibly large
- $k$ relatively small, e.g. $k = 100$ for $n = 10000$
- VERY hard problem – just getting good bounds is tough

## **3.** AC-OPF problem in rectangular coordinates

Given a power grid, determine voltages at every node so as to minimize a convex objective

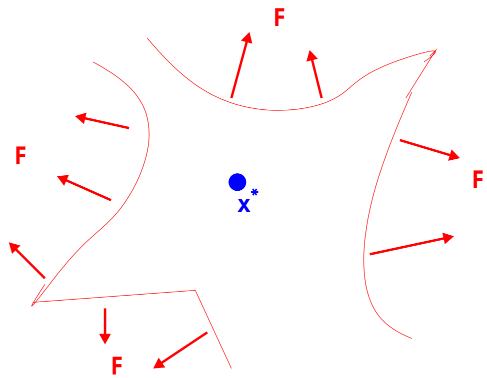
$$\begin{aligned} &\min \; v^T A v \\ &\text{s.t.} \quad L_k \le v^T F_k v \le U_k, \quad k = 1, \ldots, K \\ &\quad v \in \mathbb{R}^{2n}, \quad (n = \text{number of nodes}) \end{aligned}$$

- voltages are complex numbers; $v$ is the vector of voltages in rectangular coordinates (real and imaginary parts)
- $A \succeq 0$
- $n$ could be in the tens of thousands, or more
- the $F_k$ are very sparse (neighborhood structure for every node)
- Problem HARD when grid under distress and $L_k \approx U_k$.

## Why are these problems so hard

$$\text{Generic problem:} \quad \min\ Q(x), \quad s.t. \quad x \in F,$$

- $Q(x)$ (strongly) convex, especially: positive-definite quadratic
- $F$ nonconvex

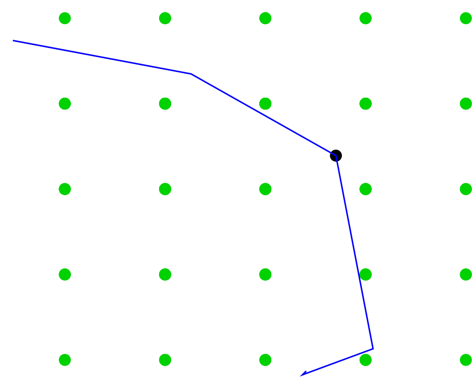



$x^*$ solves $\min \left\{ Q(x), \quad : \quad x \in \hat{F} \right\}$ where $F \subset \hat{F}$ and $\hat{F}$ convex

$\longrightarrow$ straightforward relaxations are weak

Lattice-free cuts for **linear** integer programming

Generic problem: $\min\ c^T x, \quad s.t. \quad Ax \le b, \ \ z \in Z^n$



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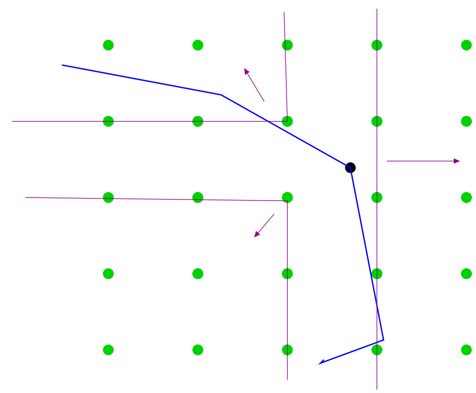

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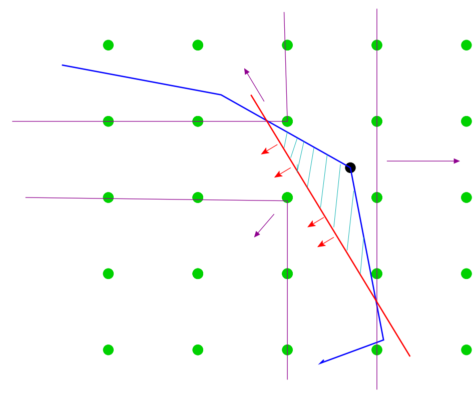

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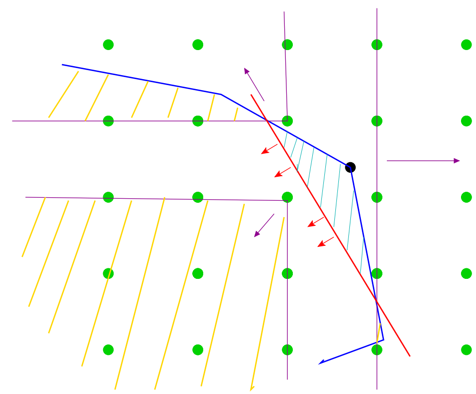

Special case: standard **disjunctions**

How to apply in a continuous, nonconvex setting?

# Exclude-and-cut

$$\begin{aligned} &\min \;\; z \\ \text{s.t.} \quad & z \geq Q(x), \\ & x \in F \end{aligned}$$

**0.** $\hat{F}$: a **convex relaxation** of $\text{conv}\,\{(x,z)\,:\, z \geq Q(x),\; x \in F\}$

**1.** Let $(x^*, z^*) \;=\; \text{argmin}\{\, z\,:(x,z) \in \hat{F}\}$

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Examples: lattice-free sets, geometry

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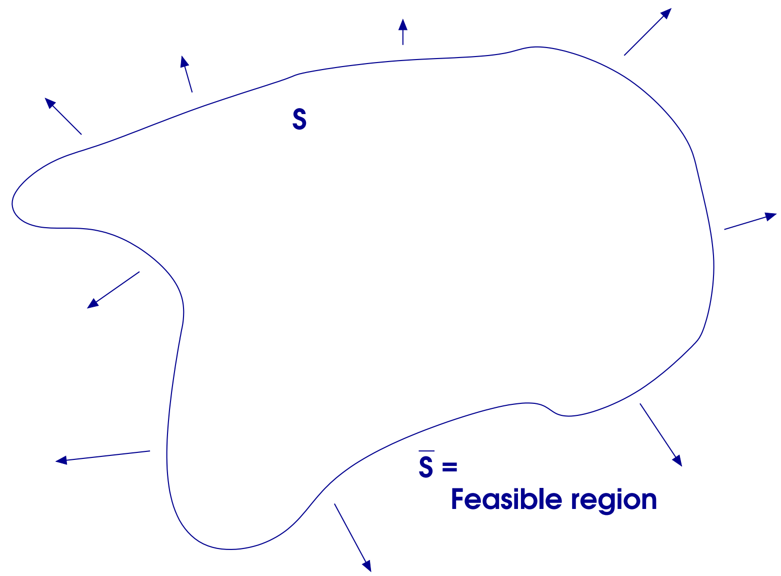
**3.** Add to the formulation an inequality $\boldsymbol{a z + \alpha^T x \geq \alpha_0}$ valid for

$$\{\, (x,z) \,:\, x \in \overline{S},\ z \geq Q(x) \,\}$$

but violated by $(x^*, z^*)$.

Valid **linear** inequalities for $\{\, (x, z) \;:\; x \in \overline{S},\;\; z \geq Q(x) \,\}$.



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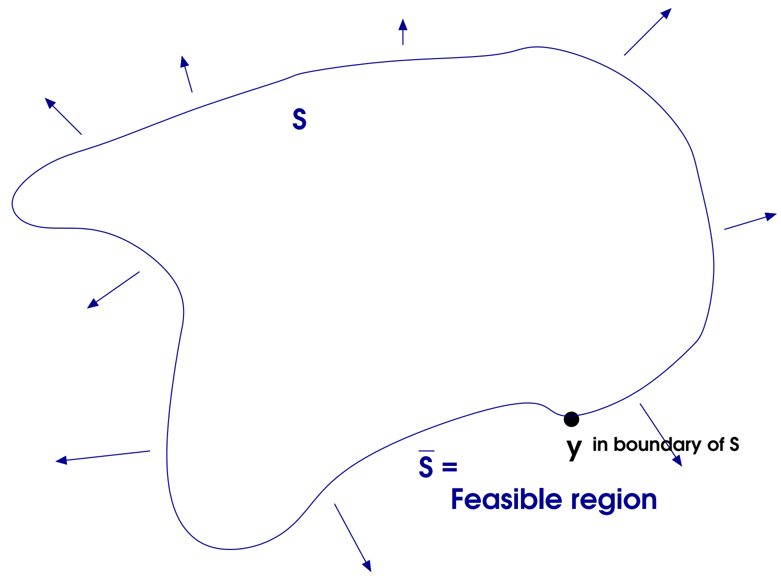


First order inequality:

$$z \;\geq\; [\nabla Q(y)]^T (x - y) + Q(y)$$

is valid EVERYWHERE – does not cut-off any points

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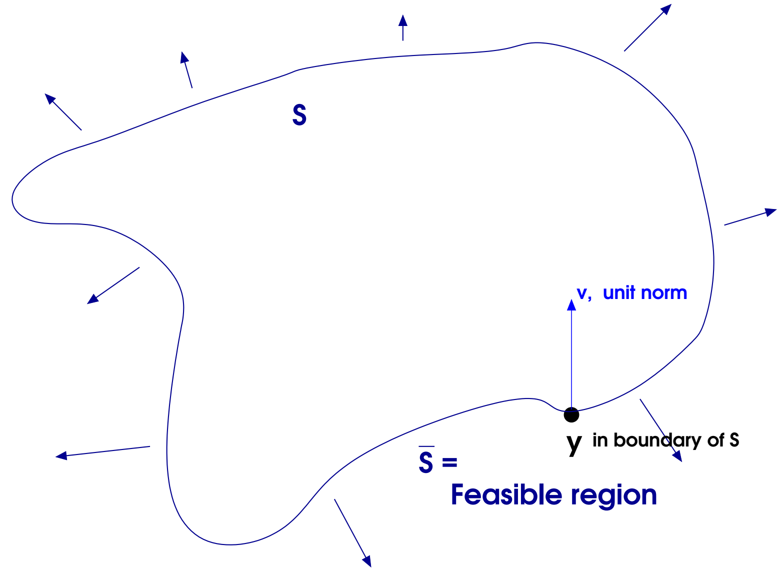


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NOT valid EVERYWHERE: RHS $> Q(x)$ for $\alpha > 0$, $v^T(x-y) > 0$ and $x \approx y$.

– want $RHS \leq Q(x)$ in $\bar{S}$ ($\alpha = 0$ always OK)

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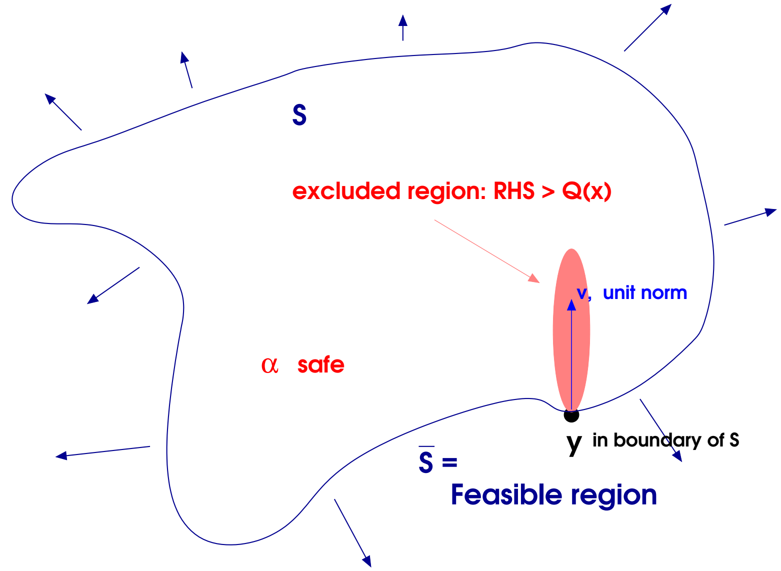


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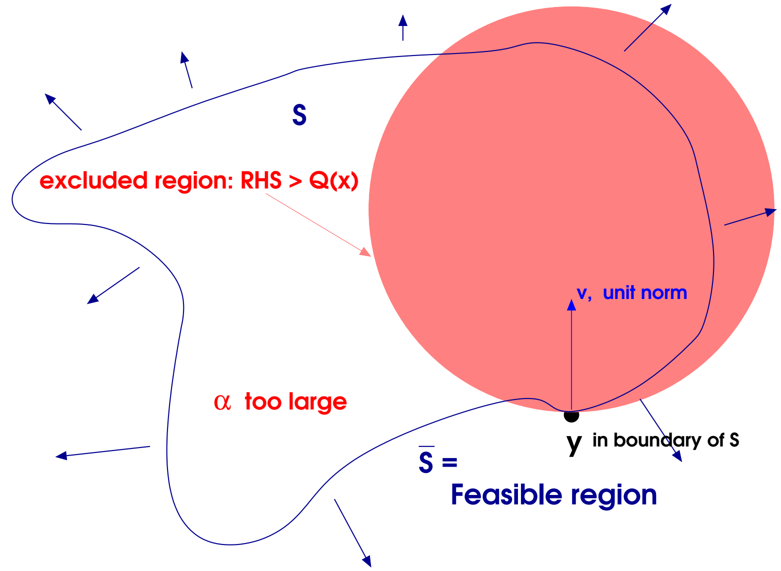


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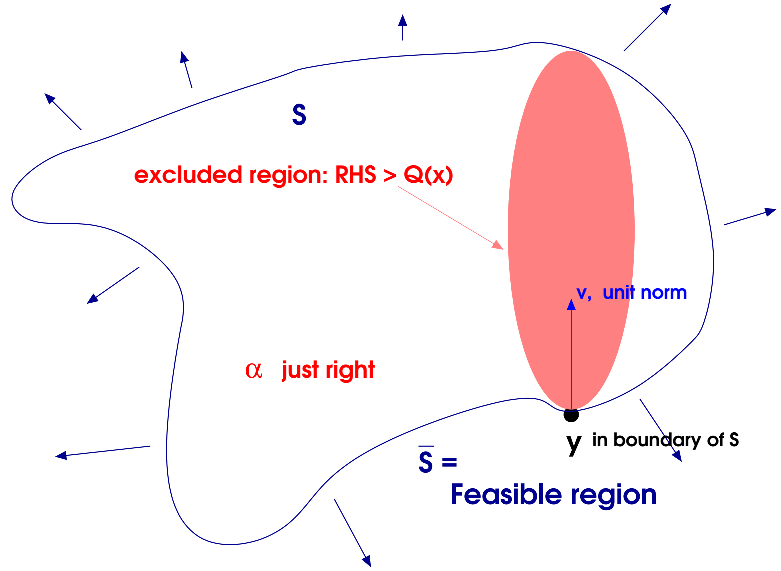


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Valid **linear** inequalities for $\mathcal{F} \doteq \{\,(x,z) \in \mathbb{R}^n \times \mathbb{R} \;:\; x \in \overline{S},\; z \geq Q(x)\,\}$.

Given $y \in \partial S$, let

$$\alpha^* \doteq \quad \boldsymbol{sup}\,\{\,\boldsymbol{\alpha \geq 0} \;:\; Q(x) \;\geq\; [\nabla Q(y)]^T(x-y) + Q(y) + \alpha v^T(x-y) \;\;\boldsymbol{\}}$$

valid for $\mathcal{F}$. Note: $\alpha^* = \alpha^*(v,y)$

**Theorem.** If $Q$ is convex and differentiable, then $conv(\mathcal{F})$ is given by

$$\begin{aligned} Q(x) \;&\geq\; [\nabla Q(y)]^T(x-y) + Q(y) \qquad \forall y \\ Q(x) \;&\geq\; [\nabla Q(y)]^T(x-y) + Q(y) + \alpha^* v^T(x-y) \\ &\qquad\qquad \forall v \text{ and } y \in \partial S. \end{aligned}$$

(abridged)

## Quadratics in action

Lifted first-order inequalities for $\mathcal{F} \; = \; \{ \, (x, z) \; : \; x \in \overline{S}, \; z \; \geq \; Q(x) \, \}$.

$$Q(x) \succ 0$$

**Separation problem.** Given $(x^*, z^*) \in \mathbb{R}^n \times \mathbb{R}$, find a lifted-first order inequality maximally violated by $(x^*, z^*)$ (if any)

**Theorem:** We can separate in polynomial time when:

- $\bar{S}$ (or $S$) is a union of polyhedra
- $S$ is an ellipsoid or paraboloid (many cases)

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**Key proof technique:** S-Lemma

$$\begin{array}{rl} \min & Q_1(x) \\ \text{s.t.} & Q_2(x) \ \leq \ 0 \\ & x \in \mathbb{R}^n \end{array}$$

($Q_i(x)$ **arbitrary** quadratics) is poly-time solvable

## S-Lemma:

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## Trust-region subproblem:

$$\begin{array}{rrcl} \min & Q_1(x) & & \\ \text{s.t.} & \|x\| & \leq & 1 \\ & & x \in \mathbb{R}^n & \end{array}$$

## Extension

$$
\begin{array}{llrcll}
\textbf{(TGEN):} & \min & x^T A x + b^T x + c & & & \\
& \text{s.t.} & \|x - x^k\|^2 & \leq & f_k & k = 1, \ldots, L_k \\
& & \|x - y^k\|^2 & \geq & g_k & k = 1, \ldots, M_k \\
& & \|x - z^k\|^2 & = & h_k & k = 1, \ldots, E_k \\
& & a_i^T x & \leq & b_i & i = 1, \ldots, m \\
& & x \in \mathbb{R}^n. & & &
\end{array}
$$

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- $P = \{x \,:\, a_i^T x \leq b_i \quad i = 1, \ldots, m\}$
- $F^* =$ the number of **faces** of $P$ that intersect $\bigcap_k \{x \,:\, \|x - x^k\| \leq f_k\}$.

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**Theorem:** For every **fixed** $L_k \geq 1$, $M_k \geq 0$, $E_k \geq 0$, problem **TGEN** can be solved in time polynomial in the problem size and $F^*$.

(SODA 2014)

Extends results by Ye, Ye-Zhang, Burer-Anstreicher, Burer-Yang

## Even more general

Barvinok (STOC 1992):

For each fixed $\boldsymbol{p \geq 1}$, there is a polynomial-time algorithm for deciding feasibility of a system

$$\begin{aligned} x^T M_i\, x &= 0, \quad 1 \leq i \leq p, \\ \|x\| &= 1, \end{aligned}$$

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where the $M_i$ are general matrices.

- **Non-constructive.** Algorithm says "yes" or "no."
- **Computational model?**

**Theorem**.

For each fixed $m \geq 1$ there is a polynomial-time algorithm that, given an optimization problem

$$\begin{aligned} \min \quad & f_0(x) \doteq x^T Q_0 x + c_0^T x \\ \text{s.t.} \quad & x^T Q_i x + c_i^T x + d_i \leq 0 \quad 1 \leq i \leq m, \end{aligned}$$

where $Q_1 \succ 0$, and $0 < \epsilon < 1$, either

**(1)** proves that the problem is infeasible,

**or**

**(2)** computes an $\epsilon$-feasible vector $\hat{x}$ such that there exists no feasible $x \in \mathbb{R}^n$ with $f_0(x) < f(\hat{x}) - \epsilon$.

The complexity of the algorithm is polynomial in the number of bits in the data and in $\log \epsilon^{-1}$