Recent results on solving QCQPs, and related topics

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Three problems

1. The “SUV” problem
   
   - given full-dimensional polyhedra $P^1, \ldots, P^K$ in $\mathbb{R}^d$,
   
   - find a point closest to the origin not contained inside any of the $P^h$.

   \[
   \min_{x \in \mathbb{R}^d} ||x||^2 \\
   \text{s.t. } x \in \mathbb{R}^d - \bigcup_{h=1}^K \text{int}(P^h),
   \]

   (application: X-ray lithography)
• Typical values for $d$ (dimension): less than 20; usually even smaller
• Typical values for $K$ (number of polyhedra): possibly hundreds, but often less than 50
• Very hard problem
2. Cardinality constrained, convex quadratic programming.

\[
\begin{align*}
\min & \quad x^T Q x + c^T x \\
\text{s.t.} & \quad Ax \leq b \\
& \quad x \geq 0, \quad \|x\|_0 \leq k
\end{align*}
\]

\(\|x\|_0 = \text{number of nonzero entries in } x\).

- \(Q \succeq 0\)
- \(x \in \mathbb{R}^n \) for \(n\) possibly large
- \(k\) relatively small, e.g. \(k = 100\) for \(n = 10000\)
- VERY hard problem – just getting good bounds is tough
3. AC-OPF problem in rectangular coordinates

Given a power grid, determine voltages at every node so as to minimize a convex objective

\[
\min v^T Av \\
\text{s.t. } L_k \leq v^T F_k v \leq U_k, \quad k = 1, \ldots, K \\
v \in \mathbb{R}^{2n}, \quad (n = \text{number of nodes})
\]

- voltages are complex numbers; \( v \) is the vector of voltages in rectangular coordinates (real and imaginary parts)
- \( A \succeq 0 \)
- \( n \) could be in the tens of thousands, or more
- the \( F_k \) are very sparse (neighborhood structure for every node)
- Problem HARD when grid under distress and \( L_k \approx U_k \).
Why are these problems so hard

Generic problem: \( \min Q(x), \quad s.t. \quad x \in F, \)

- \( Q(x) \) (strongly) convex, especially: positive-definite quadratic
- \( F \) nonconvex

\[ x^* \text{ solves } \min \left\{ Q(x), : x \in \hat{F} \right\} \text{ where } F \subset \hat{F} \text{ and } \hat{F} \text{ convex} \]

\( \rightarrow \) straightforward relaxations are weak
Lattice-free cuts for **linear** integer programming

Generic problem: \[ \min c^T x, \quad \text{s.t.} \quad Ax \leq b, \quad z \in \mathbb{Z}^n \]
Lattice-free cuts for **linear** integer programming

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Special case: standard **disjunctions**

How to apply in a continuous, nonconvex setting?
Exclude-and-cut

\[
\begin{align*}
\min \ z \\
\text{s.t.} \quad z &\geq Q(x), \\
x &\in F
\end{align*}
\]

0. \( \hat{F} \): a convex relaxation of \( \text{conv} \ \{(x, z) : z \geq Q(x), \ x \in F\} \)

1. Let \( (x^*, z^*) = \arg\min \{z : (x, z) \in \hat{F}\} \)
Exclude-and-cut

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0. $\hat{F}$: a convex relaxation of $\text{conv} \{(x, z) : z \geq Q(x), \ x \in F\}$

1. Let \((x^*, z^*) = \text{argmin}\{z : (x, z) \in \hat{F}\}\)

2. Find an open set \(S\) s.t. \(x^* \in S\) and \(S \cap F = \emptyset\).
   Examples: lattice-free sets, geometry
Exclude-and-cut

\[\begin{align*}
\text{min} & \quad z \\
\text{s.t.} & \quad z \geq Q(x), \\
& \quad x \in F
\end{align*}\]

0. \(\hat{F}\): a \textbf{convex relaxation} of \(\text{conv} \{ (x, z) : z \geq Q(x), \ x \in F \}\)

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3. Add to the formulation an inequality \(az + \alpha^T x \geq \alpha_0\) valid for

   \[\{(x, z) : x \in \overline{S}, \ z \geq Q(x)\}\]

   but violated by \((x^*, z^*)\).
Valid **linear** inequalities for \( \{(x, z) : x \in \overline{S}, z \geq Q(x)\} \).
Valid **linear** inequalities for \( \{(x, z) : x \in \overline{S}, \ z \geq Q(x)\} \).

\[ z \geq [\nabla Q(y)]^T (x - y) + Q(y) \]

is valid EVERYWHERE – does not cut-off any points
Valid **linear** inequalities for \( \{ (x, z) : x \in \overline{S}, z \geq Q(x) \} \).

First order inequality:

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z \geq \left[ \nabla Q(y) \right]^T (x - y) + Q(y)
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is valid EVERYWHERE – does not cut-off any points **Lifted** first order inequality, for \( \alpha \geq 0 \):

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z \geq \left[ \nabla Q(y) \right]^T (x - y) + Q(y) + \alpha v^T (x - y)
\]

NOT valid EVERYWHERE: RHS \( > Q(x) \) for \( \alpha > 0 \), \( v^T (x - y) > 0 \) and \( x \approx y \).

- want \( RHS \leq Q(x) \) in \( \tilde{S} \) (\( \alpha = 0 \) always OK)
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**first-order term** \( \approx Q(x) \) **lifting**

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Valid **linear** inequalities for \( \mathcal{F} \doteq \{ (x, z) \in \mathbb{R}^n \times \mathbb{R} \mid x \in \overline{S}, \ z \geq Q(x) \} \).

Given \( y \in \partial S \), let
\[
\alpha^* \doteq \sup \{ \alpha \geq 0 : Q(x) \geq [\nabla Q(y)]^T(x - y) + Q(y) + \alpha v^T(x - y) \}
\]
valid for \( \mathcal{F} \). Note: \( \alpha^* = \alpha^*(v, y) \)

**Theorem.** If \( Q \) is convex and differentiable, then \( \text{conv}(\mathcal{F}) \) is given by
\[
Q(x) \geq [\nabla Q(y)]^T(x - y) + Q(y) \quad \forall y
\]
\[
Q(x) \geq [\nabla Q(y)]^T(x - y) + Q(y) + \alpha^* v^T(x - y) \quad \forall v \text{ and } y \in \partial S.
\]

(abridged)
Quadratics in action

Lifted first-order inequalities for $\mathcal{F} = \{(x, z) : x \in \bar{S}, z \geq Q(x)\}$.

$Q(x) \succ 0$

Separation problem. Given $(x^*, z^*) \in \mathbb{R}^n \times \mathbb{R}$, find a lifted-first order inequality maximally violated by $(x^*, z^*)$ (if any)

Theorem: We can separate in polynomial time when:

- $\bar{S}$ (or $S$) is a union of polyhedra
- $S$ is an ellipsoid or paraboloid (many cases)
Quadratics in action

Lifted first-order inequalities for \( \mathcal{F} = \{(x, z) : x \in \overline{S}, \ z \geq Q(x)\} \).

\[ Q(x) \geq 0 \]

**Separation problem.** Given \((x^*, z^*) \in \mathbb{R}^n \times \mathbb{R}, \) find a lifted-first order inequality maximally violated by \((x^*, z^*)\) (if any)

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\[ Q(x) \geq 0 \]

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Key proof technique: S-Lemma

\[
\begin{align*}
\min & \quad Q_1(x) \\
\text{s.t.} & \quad Q_2(x) \leq 0 \\
& \quad x \in \mathbb{R}^n
\end{align*}
\]

\((Q_i(x)\) arbitrary quadratics) is poly-time solvable
S-Lemma:

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\((Q_i(x) \text{ arbitrary quadratics})\) is poly-time solvable
Trust-region subproblem:

\[
\begin{align*}
\min & \quad Q_1(x) \\
\text{s.t.} & \quad \|x\| \leq 1 \\
& \quad x \in \mathbb{R}^n
\end{align*}
\]
(TGEN): \[
\min \quad x^T Ax + b^T x + c \\
\text{s.t.} \quad \|x - x^k\|^2 \leq f_k \quad k = 1, \ldots, L_k \\
\|x - y^k\|^2 \geq g_k \quad k = 1, \ldots, M_k \\
\|x - z^k\|^2 = h_k \quad k = 1, \ldots, E_k \\
a^T_i x \leq b_i \quad i = 1, \ldots, m \\
x \in \mathbb{R}^n.
\]
Extension

(TGEN): \[ \begin{align*}
\text{min} & \quad x^T Ax + b^T x + c \\
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& \quad \|x - z^k\|^2 = h_k \quad k = 1, \ldots, E_k \\
& \quad a_i^T x \leq b_i \quad i = 1, \ldots, m \\
& \quad x \in \mathbb{R}^n.
\end{align*} \]

\begin{itemize}
\item \( P = \{x : a_i^T x \leq b_i \quad i = 1, \ldots, m\} \)
\item \( F^* = \) the number of faces of \( P \) that intersect \( \bigcap_k \{x : \|x - x^k\| \leq f_k\} \).
\end{itemize}
\textbf{Extension}

\textbf{(TGEN)}: \quad \min \quad x^T A x + b^T x + c
\text{s.t.} \quad \|x - x^k\|^2 \leq f_k \quad k = 1, \ldots, L_k
\|x - y^k\|^2 \geq g_k \quad k = 1, \ldots, M_k
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a^T_i x \leq b_i \quad i = 1, \ldots, m
x \in \mathbb{R}^n.

- \( P = \{ x : a^T_i x \leq b_i \quad i = 1, \ldots, m \} \)
- \( F^* = \) the number of faces of \( P \) that intersect \( \bigcap_k \{ x : \|x - x^k\| \leq f_k \} \).

\textbf{Theorem:} For every fixed \( L_k \geq 1, M_k \geq 0, E_k \geq 0 \), problem \( \text{TGEN} \) can be solved in time polynomial in the problem size and \( F^* \).

\textit{(SODA 2014)}

Extends results by Ye, Ye-Zhang, Burer-Anstreicher, Burer-Yang
Even more general

Barvinok (STOC 1992):

For each fixed \( p \geq 1 \), there is a polynomial-time algorithm for deciding feasibility of a system

\[
   x^T M_i x = 0, \quad 1 \leq i \leq p, \\
   \|x\| = 1,
\]

where the \( M_i \) are general matrices.
Even more general

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where the \( M_i \) are general matrices.

- **Non-constructive.** Algorithm says “yes” or “no.”

- **Computational model?**
Theorem.

For each fixed $m \geq 1$ there is a polynomial-time algorithm that, given an optimization problem

$$\min \ f_0(x) = x^T Q_0 x + c_0^T x$$

s.t. $$x^T Q_i x + c_i^T x + d_i \leq 0 \quad 1 \leq i \leq m,$$

where $Q_1 \succ 0$, and $0 < \epsilon < 1$, either

(1) proves that the problem is infeasible,

or

(2) computes an $\epsilon$-feasible vector $\hat{x}$ such that there exists no feasible $x \in \mathbb{R}^n$ with $f_0(x) < f(\hat{x}) - \epsilon$.

The complexity of the algorithm is polynomial in the number of bits in the data and in $\log \epsilon^{-1}$