CUT GENERATING FUNCTIONS

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Mixed Integer Linear Programming

$$
\begin{array}{lll}
\min & cx & \\
\text{s.t.} & Ax = b & \\
 & x_j \in \mathbb{Z} & \text{for } j = 1, \ldots, p \\
 & x_j \geq 0 & \text{for } j = 1, \ldots, n.
\end{array}
$$

Cutting plane approach to solving MILP:

- First solve the LP relaxation. Basic optimal solution:

$$
x_i \quad = \quad f_i + \sum_{j \in N} r^j x_j \qquad \text{for } i \in B.
$$

- If $f_i \notin \mathbb{Z}$ for some $i \in B \cap \{1, \ldots, p\}$, add cutting planes.

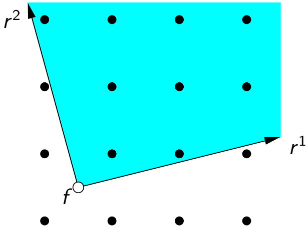
Corner Relaxation

Gomory 1969: Relax nonnegativity on basic variables.

In addition, Andersen, Louveaux, Weismantel and Wolsey 2007 suggested to relax integrality on the nonbasic variables x_j.

$$\begin{array}{rcl} y & = & f + \sum_{j=1}^{k} r^j x_j \\ y & \in & \mathbb{Z}^q \\ x & \geq & 0 \end{array}$$

Example



Feasible set $\left\{ \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \in \mathbb{Z}^2 : \right.$

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = f + r^1 x_1 + r^2 x_2$$

$\left. \text{where} \quad x_1 \geq 0, x_2 \geq 0 \right\}$

Formulas for Cutting Planes

$$\begin{array}{rcl} y & = & f + \sum_{j=1}^{k} r^j x_j \\ y & \in & \mathbb{Z}^q \\ x & \geq & 0 \end{array}$$

Every inequality cutting off the point $(\bar{x}, \bar{y}) = (0, f)$ is of the form $\sum_{j=1}^{k} \alpha_j x_j \geq 1$.

We are interested in "formulas" for deriving such inequalities.

More formally, we are interested in functions $\psi : \mathbb{R}^q \to \mathbb{R}$ such that the inequality

$$\sum_{j=1}^{k} \psi(r^j) x_j \geq 1$$

is valid for every choice of k and vectors $r^1, \ldots, r^k \in \mathbb{R}^q$.

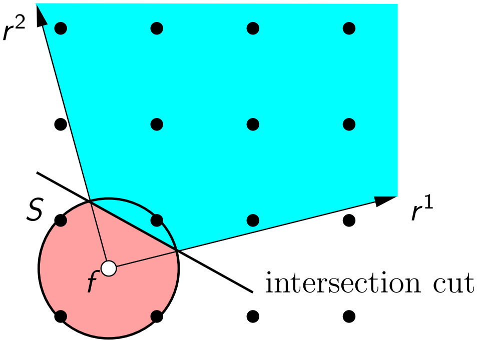
We refer to such functions ψ as cut-generating functions.

We are interested in minimal cut-generating functions.

Intersection Cuts

Balas 1971

Assume $f \notin \mathbb{Z}^q$. Want to cut off the basic solution $x = 0, y = f$.



Any convex set S with $f \in int(S)$ with no integer point in $int(S)$.

The gauge of $S - f$, i.e. $\psi(r) = inf\{\lambda > 0 : \frac{1}{\lambda} r \in S - f\}$, is a cut-generating function.

Intersection cut: $\psi(r^1)x_1 + \psi(r^2)x_2 \geq 1$.

Theorem

Borozan and Cornuéjols (MOR 2009)

Let $f \in \mathbb{R}^q \setminus \mathbb{Z}^q$.

If $\psi : \mathbb{R}^q \to \mathbb{R}$ is a minimal cut-generating function, then ψ is

- nonnegative
- piecewise linear
- positively homogeneous
- and convex.

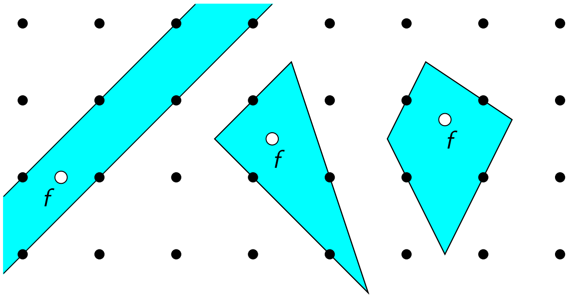
Furthermore $B_\psi := \{y \in \mathbb{R}^q : \ \psi(y - f) \leq 1\}$ is a maximal $\mathbb{Z}^q$-free convex set containing f in its interior.

Conversely, for any maximal $\mathbb{Z}^q$-free convex set B containing f in its interior, the gauge of $B - f$ is a minimal cut-generating function ψ.

DEFINITION A convex set is $\mathbb{Z}^q$-free if it does not have any integral point in its interior. However, it may have integral points on its boundary.

Maximal $\mathbb{Z}^q$-Free Sets in the Plane

Split, triangles and quadrilaterals



generate split, triangle and quadrilateral inequalities $\sum \psi(r) x_r \geq 1$, where the function ψ is the gauge of $S - f$.

If $S = \{y \in \mathbb{R}^q : a_i(y - f) \leq 1, i = 1, \dots, t\}$,
then $\psi(r) = max_{i=1,\dots,t}\; a_i r$.

Integer Lifting

Here, we consider a system of the form

$$\begin{array}{rcl} y & = & f + \sum_{j=1}^{k} r^j x_j + \sum_{i=1}^{\ell} \rho^i z_i \\ y & \in & \mathbb{Z}^q \\ x & \geq & 0 \\ z_i & \in & \mathbb{Z},\ i = 1, \ldots, \ell. \end{array}$$

We are interested in functions $\psi : \mathbb{R}^q \to \mathbb{R}$ and $\pi : \mathbb{R}^q \to \mathbb{R}$ such that the inequality

$$\sum_{j=1}^{k} \psi(r^j) x_j + \sum_{i=1}^{\ell} \pi(\rho^i) z_i \geq 1$$

is valid for every choice of integers k, ℓ and vectors $r^1, \ldots, r^k \in \mathbb{R}^q$ and $\rho^1, \ldots, \rho^\ell \in \mathbb{R}^q$.

Gomory and Johnson since the 1970's: Construct π first, then ψ.

Integer Lifting

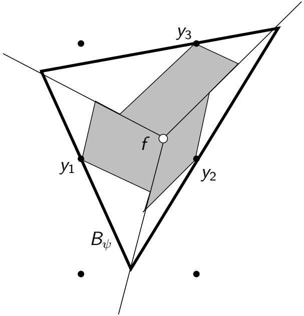
Dey and Wolsey (IPCO 2008)

Starting from a minimal cut-generating function $\psi : \mathbb{R}^q \to \mathbb{R}$, what can we say about a minimal lifting function π?

Clearly, $\pi \leq \psi$. Is there a region R where we can guarantee that $\pi(r) = \psi(r)$ for all $r \in R$? The answer is YES.

Basu, Campelo, Conforti, Cornuéjols, Zambelli (Math Prog 2013):

THEOREM Region where $\pi = \psi$: $R = \bigcup_t R(y_t)$ where the union is taken over all integral points y_t on the boundary of the maximal $\mathbb{Z}^q$-free convex set B_ψ defining ψ and the $R(y_t)$s are parallelepipeds as shown in grey in the figure.



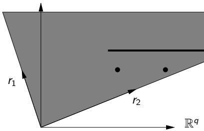
Conforti, Cornuéjols, Daniilidis, Lemaréchal, Malick 2014

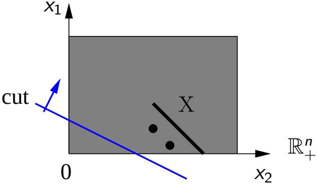
deal with sets of the form

$$X := \{x \in \mathbb{R}^n_+ \; : \; Rx \in S\}$$

where

$$R = [r_1, \ldots, r_n] \text{ is a real } q \times n \text{ matrix,}$$
$$S \subset \mathbb{R}^q \text{ is a nonempty closed set with } 0 \notin S.$$





Since $0 \notin S$, the closed convex hull of X does not contain 0.
We are interested in *separating* 0 from X, which we write as

$$c^\top x \geqslant 1 \, , \quad \text{for all } x \in X.$$

Motivation arising in mixed integer programming

Start from a polyhedron

$$P = \left\{(x, y) \in \mathbb{R}^n_+ \times \mathbb{R}^q \;:\; Ax + y = b\,\right\}$$

and assume that $b \notin \mathbb{Z}^q$.

Example 1   Andersen, Louveaux, Weismantel and Wolsey 2007

The set of interest is $P \cap \{\mathbb{R}^n_+ \times \mathbb{Z}^q\}$,
I.e. we want $(x, y = b - Ax)$ such that $x \in \mathbb{R}^n_+$ and $b - Ax \in \mathbb{Z}^q$.

This fits our model by taking

$$R = -A\,, \quad S = \mathbb{Z}^q - b$$

Motivation arising from complementary slackness

Example 2 Still using

$$P = \left\{(x, y) \in \mathbb{R}^n_+ \times \mathbb{R}^q \ : \ Ax + y = b \right\}$$

let

$$E \subset \{1, 2, \ldots, q\} \times \{1, 2, \ldots, q\}$$

and

$$C := \{y \in \mathbb{R}^q_+ \ : \ y_i y_j = 0\,, (i, j) \in E\}.$$

The set of interest is then $P \cap (\mathbb{R}^n_+ \times C)$.

It can be modeled in our framework where

$$R = -A\,, \quad S = C - b;$$

Cuts have been used for complementarity problems of this type, for example in Judice, Sherali, Ribeiro, Faustino 2006

Cut-generating functions

We will retain from the above examples the asymmetry between S – a very particular and highly structured set – and R – an arbitrary matrix. Keeping this in mind, we will consider that (q, S) *is given and fixed*, while (n, R) *is instance-dependent data*.

Let S be fixed. Consider a function

$$\rho \ : \ \mathbb{R}^q \mapsto \mathbb{R}$$

that produces coefficients $c_j := \rho(r_j)$ of a cut $c^\top x \geqslant 1$ valid for X for any choice of n and $R = [r_1 \dots r_n]$.

In summary, we require our ρ to satisfy

$$\forall R = [r_1 \dots r_n], \quad x \in X \quad \Longrightarrow \quad \sum_{j=1}^{n} \rho(r_j) x_j \geqslant 1.$$

Such a ρ can then justifiably be called a *cut-generating function*.

Sufficiency of cut-generating functions

Cut-generating functions are defined assuming that S is fixed but R can vary arbitrarily.

What happens if both S and R are fixed?
A natural question is whether, for every cut $c^\top x \geqslant 1$ that is valid for X, there exists some cut-generating function ρ such that $\rho(r_j) \leq c_j$.

THEOREM Cornuejols, Wolsey, Yildiz (Math Prog 2014)
Suppose $S \subset \text{cone}(R)$. Then any valid inequality $c^\top x \geq 1$ separating 0 from X is dominated by one obtained from a cut-generating function.

Next we show that the (vast!) class of cut-generating functions from $\mathbb{R}^q$ to $\mathbb{R}$ can be drastically reduced.

Cut-generating functions

Let $\bar{\rho}(r) := \inf_{K,\alpha}\left\{\sum_{k=1}^{K} \alpha_k \rho(r_k) \;:\; \sum_{k=1}^{K} \alpha_k r_k = r,\, \alpha_k \geqslant 0\right\}.$

THEOREM

If ρ is a cut-generating function, then $\bar{\rho}$ is nowhere $-\infty$ and is again a cut-generating function.

The function $\bar{\rho}$ is *sublinear* (convex and positively homogeneous). Sublinear functions are continuous.
Because $\bar{\rho} \leq \rho$, the theorem shows that sublinear functions suffice to generate all relevant cuts; a fairly narrow class indeed, which is fundamental in convex analysis.

Sublinear functions are in correspondence with closed convex sets and in our context, such a correspondence is based on the mapping $\rho \mapsto V$ defined by

$$V := \left\{r \in \mathbb{R}^q \;:\; \rho(r) \leqslant 1\right\}.$$

S-free sets

The set V turns out to be a cornerstone: the theorem below establishes a correspondence between cut-generating functions and the so-called *S-free sets*.

DEFINITION

Given a closed set $S \subset \mathbb{R}^q$ not containing the origin, a closed convex neighborhood V of $0 \in \mathbb{R}^q$ is called S-free if its interior contains no point in S.

THEOREM

Let ρ be a sublinear function and $V := \{r \in \mathbb{R}^q : \rho(r) \leqslant 1\}$. Then ρ is a cut-generating function if and only if V is S-free.

Representation

As a result, cut-generating functions can alternatively be studied from a geometric point of view, involving sets V instead of functions ρ. This situation, common in convex analysis, is often very fruitful. However, there is a difficulty here: the mapping $\rho \mapsto V$ is many-to-one and therefore has no inverse.

DEFINITION
Let $V \subset \mathbb{R}^q$ be a closed convex neighborhood of the origin. A *representation* of V is a sublinear function ρ satisfying $V = \left\{r \in \mathbb{R}^q \,:\, \rho(r) \leqslant 1\right\}$.

A cut-generating function is a representation of an S-free set. Among the several representations of an S-free set V, we are interested in the small ones.

Minimal representation

The following geometric object turns out to be relevant:

$\hat{V}^\circ := \left\{ d \in \mathbb{R}^q : \sup_{r \in V} d^\top r = 1 \right\}$. Let $\mu_V(r) := \sup_{d \in \hat{V}^\circ} d^\top r$.

PROPOSITION Basu, Cornuéjols and Zambelli (JOCA 2011)

Any sublinear function ρ representing V satisfies $\rho \geqslant \mu_V$.

Let γ_V denote Minkowski's gauge function.

THEOREM

A sublinear function ρ represents V if and only if it satisfies

$$\mu_V \leqslant \rho \leqslant \gamma_V.$$