Integer Quadratic Programming in the Plane

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June 2, 2014

Integer Polynomial Programming

Integer Polynomial Programming

$$\min\{f_0^d(x) \,:\, f_i^d(x) \le 0,\, i \in I,\, x \in \mathbb{Z}^n\}$$

f_i^d is a polynomial of degree at most d

	n=1	n=2	n=9	n=58	n fixed	n general
d=1	P	P	P	P	P[a]	NPH[b]
d=2	P	NPH[c]	NPH	Und[d]	Und	Und
d=4	P	NPH	NPH	Und	Und	Und
$d=1.6 \cdot 10^{45}$	P	NPH	Und[d]	Und	Und	Und

[a] Lenstra '83

[b] Cook '71

[c] Manders & Adleman '78

[d] Matiyasevich '77, Jones '82 (Hilbert's 10^{th} problem, 1900)

Integer Polynomial Programming

- Natural extension of Integer Linear Programming
- Huge modeling power
- Wide open field

Hydropower plants

Gas pipeline transportation

Polynomial objective function

Polynomial objective function

$$\min\{f^d(x) \;:\; x \in P \cap \mathbb{Z}^n\}$$

f^d is a polynomial of degree at most d

	$n = 1$	$n = 2$	$n = 58$	n fixed	n general
$d = 1$	P	P	P	P[a]	NPH[b]
$d = 2$	P	?	?	?	NPH
$d = 3$	P	?	?	?	NPH
$d = 4$	P	NPH[c]	Und[d]	Und	Und

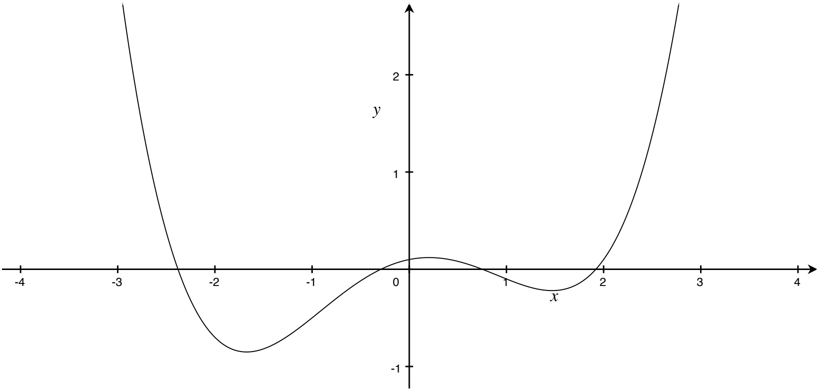
[a] Lenstra '83

[b] Cook '71

[c] De Loera, Hemmecke, Köppe & Weismantel '06

[d] Matiyasevich '77, Jones '82

One dimensional problem



- compute the zeroes of the derivative of f
  and the boundary of P
- evaluate the integer points closest to each such point
- pick the best feasible

Quartic in the plane: NP-hardness

Theorem (De Loera, Hemmecke, Köppe & Weismantel '06)

$\min\{f^4(x) \,:\, x \in P \cap \mathbb{Z}^2\}$ is NP-hard

Given $a, b, c \in \mathbb{Z}^+$, the following problem is NP-hard (Manders and Adleman, 1978):

$$\exists x \in \mathbb{Z}^+ \text{ with } x < c \text{ such that } x^2 \equiv_b a$$

$$\iff \exists x \in \mathbb{Z}^+, y \in \mathbb{Z} \text{ with } x < c \text{ such that } x^2 - a - by = 0$$

$$\iff \begin{array}{l} \min (x^2 - a - by)^2 = 0 \\ \text{s.t. } (x, y) \in \mathbb{Z}^2 \\ \qquad 1 \leq x \leq c - 1 \\ \qquad \frac{1-a}{b} \leq y \leq \frac{(c-1)^2 - a}{b} \end{array}$$

Integer Polynomial Programming

Polynomial objective function

$$\min\{f^d(x) \; : \; x \in P \cap \mathbb{Z}^n\}$$

f^d is a polynomial of degree at most d

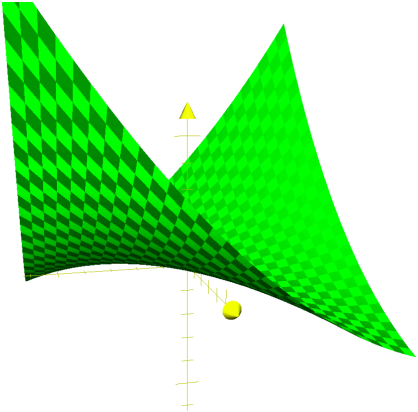
	$n=1$	$n=2$	$n=58$	n fixed	n general
$d=1$	P	P	P	P	NPH
$d=2$	P	P	?	?	NPH
$d=3$	P	?	?	?	NPH
$d=4$	P	NPH	Und	Und	Und

Integer Quadratic Programming in the plane

Theorem (DP & Weismantel '14)

$$\min\{f^2(x) \,:\, x \in P \cap \mathbb{Z}^2\}$$

can be solved in polynomial time



Integer Quadratic Programming and Number Theory

The Pell equation

$$x^2 = dy^2 + 1$$

to be solved in $x, y \in \mathbb{Z}^+$ for a given non-square $d \in \mathbb{Z}^+$

The Pell equation always has a solution, but all its solutions can be of size exponential in the size of d

The Pell equation – bounded version

$$x^2 = dy^2 + 1$$
$$x \leq u$$

to be solved in $x, y \in \mathbb{Z}^+$ for given $d, u \in \mathbb{Z}^+$

Integer Quadratic Programming in the plane

Integer Quadratic Programming in the plane (IQP)

$$\begin{aligned} \min \quad & ax^2 + bxy + cy^2 + dx + ey \\ \text{s.t.} \quad & (x, y) \in P \cap \mathbb{Z}^2 \end{aligned}$$

- integer coefficients
- P is a rational polyhedron in $\mathbb{R}^2$

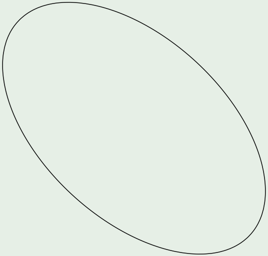
Conic sections

The conic section

$$ax^2 + bxy + cy^2 + dx + ey = \gamma$$

can be classified with the discriminant $\Delta = b^2 - 4ac$

- If $\Delta < 0$, the equation represents an ellipse



Conic sections

The conic section

$$ax^2 + bxy + cy^2 + dx + ey = \gamma$$

can be classified with the discriminant $\Delta = b^2 - 4ac$

- If $\Delta = 0$, the equation represents a parabola

Conic sections

The conic section

$$ax^2 + bxy + cy^2 + dx + ey = \gamma$$

can be classified with the discriminant $\Delta = b^2 - 4ac$

- If $\Delta > 0$, the equation represents a hyperbola

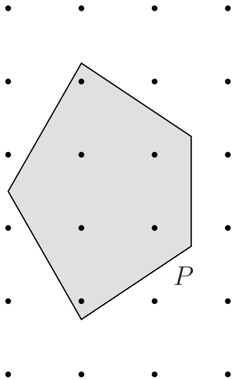
Ingredients – Concave Lemma

Concave Lemma

Let P be a rational polytope, and let f be quasi-concave on P_I. Then IQP can be solved in polynomial time.

Proof

- We can find the set V of the vertices of P_I (Hartman, 1989)
- Let $\bar{z}$ be the best vertex, and let $W := \{z \in P : f(z) \geq f(\bar{z})\}$
- $V \subseteq W$
- As W is convex, $P_I = \operatorname{conv} V \subseteq W$



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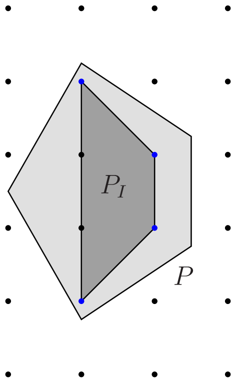

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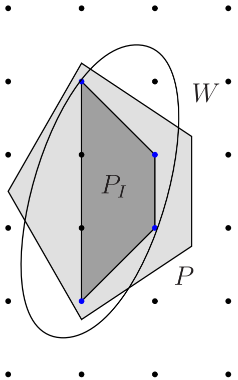


Ingredients – Khachiyan & Porkolab '00

Theorem

Let Y be a convex set defined by polynomial inequalities with integer coefficients. The feasibility problem for $Y \cap \mathbb{Z}^n$ can be solved in polynomial time.

Corollary

Let $g^i(z)$, $i = 0, \ldots, m$ be quasi-convex polynomials with integer coefficients. Then the following can be solved in polynomial time:

$$\begin{aligned} \min \quad & g_0(z) \\ s.t. \quad & g_i(z) \leq 0,\ i = 1, \ldots, m \\ & z \in \mathbb{Z}^n \end{aligned}$$

Proof - case $\Delta \leq 0$

$$\min\{ax^2 + bxy + cy^2 + dx + ey \ : \ (x, y) \in P \cap \mathbb{Z}^2\}$$

Case $a = c = 0$

- As $b^2 \leq 4ac$, it follows that $b = 0$. Then f is linear
- We can solve IQP in polynomial time (Lenstra, 1983)

Case a and c are not both zero, wlog $a \neq 0$

$$f(x, y) = a\left(x + \frac{b}{2a}y\right)^2 + \frac{-\Delta}{4a}y^2 + dx + ey$$

- If $a > 0$, then $-\Delta/4a \geq 0$. Hence f is convex. We can solve IQP in polynomial time (Khachiyan and Porkolab, 2000)
- If $a < 0$, then $-\Delta/4a \leq 0$. Thus f is concave. We can solve IQP in polynomial time (Concave Lemma)

Proof - case $\Delta > 0$

From now on we consider the case $a > 0$.

f can be rewritten as follows:

$$f(x,y) = a\Big(x + \frac{b}{2a}y + \frac{d}{2a}\Big)^2 - \frac{\Delta}{4a}\Big(y + \frac{bd-2ae}{\Delta}\Big)^2 + \frac{ae^2+cd^2-bde}{\Delta}$$

The conic section $f(x,y) = \bar{\gamma}$ is degenerate for

$$\bar{\gamma} = \frac{ae^2+cd^2-bde}{\Delta}$$

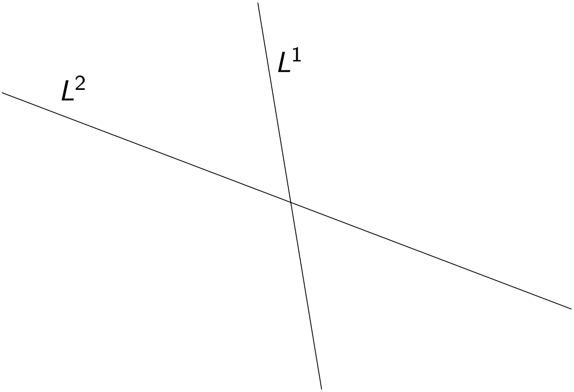
$f(x,y) = \bar{\gamma}$ if and only if

$$a\Big(x + \frac{b}{2a}y + \frac{d}{2a}\Big)^2 = \frac{\Delta}{4a}\Big(y + \frac{bd-2ae}{\Delta}\Big)^2$$

Proof

Claim

$f(x, y) = \bar{\gamma}$ for the points in $L^1 \cup L^2$.
L^1 and L^2 are not necessarily rational.

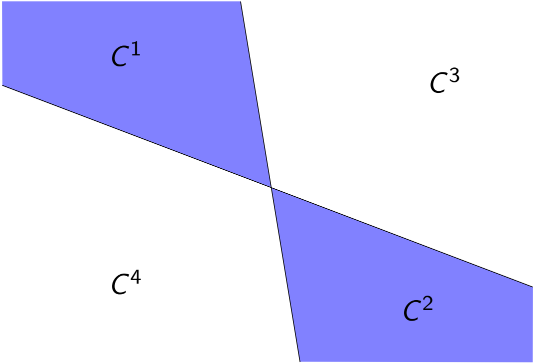


Proof

L^1 and L^2 subdivide $\mathbb{R}^2$ into four translated cones C^1, C^2, C^3, C^4.

Claim

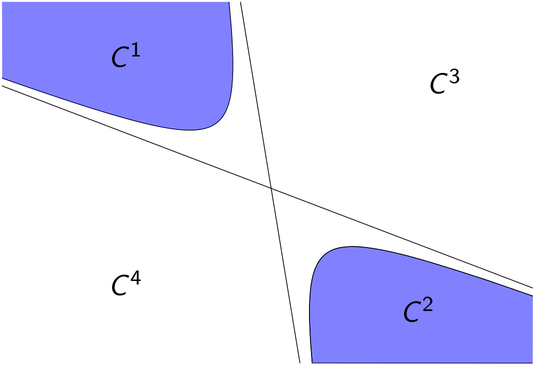
$f(x, y) \leq \bar{\gamma}$ for the points in $C^1 \cup C^2$



Proof

Claim

For every $\gamma \leq \bar{\gamma}$, the set of points that satisfy $f(x, y) \leq \gamma$ is the union of two convex sets, one contained in C^1, and the other contained in C^2.

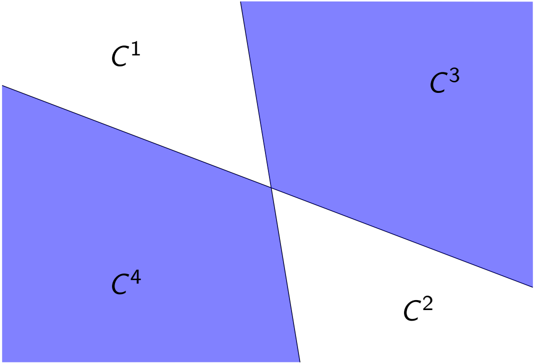


f is quasi-convex on C^1 and C^2.

Proof

Claim

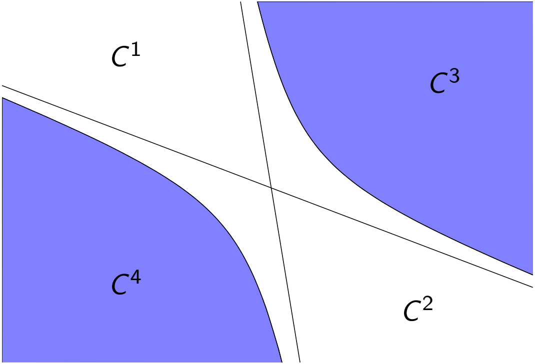
$f(x, y) \geq \bar{\gamma}$ for the points in $C^3 \cup C^4$.



Proof

Claim

For every $\gamma \geq \bar{\gamma}$, the set of points that satisfy $f(x, y) \geq \gamma$ is the union of two convex sets, one contained in C^3, and the other contained in C^4.



f is quasi-concave on C^3 and C^4.

Proof - algorithm

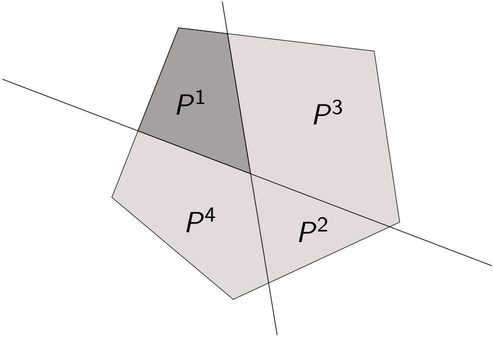
Algorithm

Let $P^i := P \cap C^i$, $i = 1, \ldots, 4$.

- Solve IQP over P^1 and over P^2.

If $(P^1 \cup P^2) \cap \mathbb{Z}^2 \neq \emptyset$, we are done. Otherwise

- solve IQP over P^3 and over P^4.



Proof - algorithm

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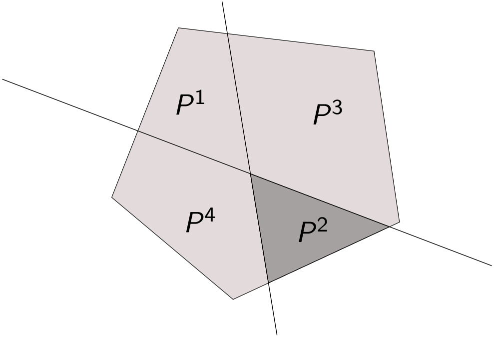

Proof - algorithm

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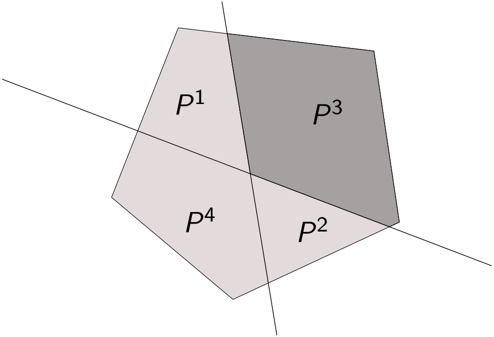


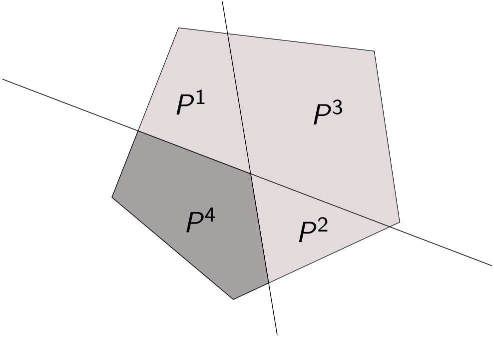
Algorithm

Let $P^i := P \cap C^i$, $i = 1, \dots, 4$.

- Solve IQP over P^1 and over P^2.

If $(P^1 \cup P^2) \cap \mathbb{Z}^2 \neq \emptyset$, we are done. Otherwise

- solve IQP over P^3 and over P^4.



Proof - region P^1

Idea

Use Khachiyan & Porkolab's algorithm in a binary search setting.

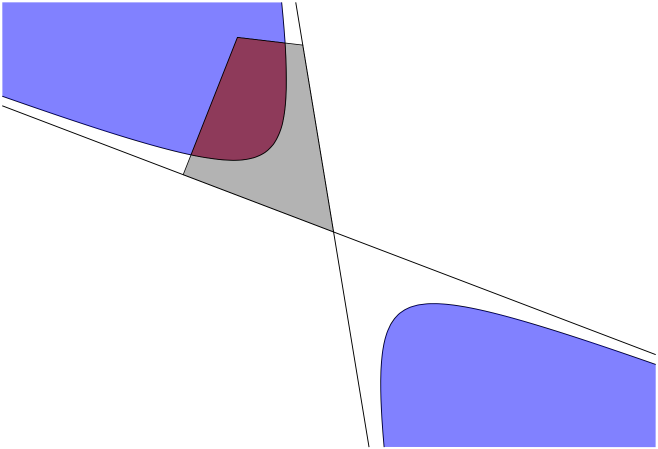
Theorem

Let Y be a convex set defined by polynomial inequalities with integer coefficients. The feasibility problem for $Y \cap \mathbb{Z}^n$ can be solved in polynomial time.

Proof - region P^1 - feasibility problem

Feasibility problem for $Y \cap \mathbb{Z}^2$

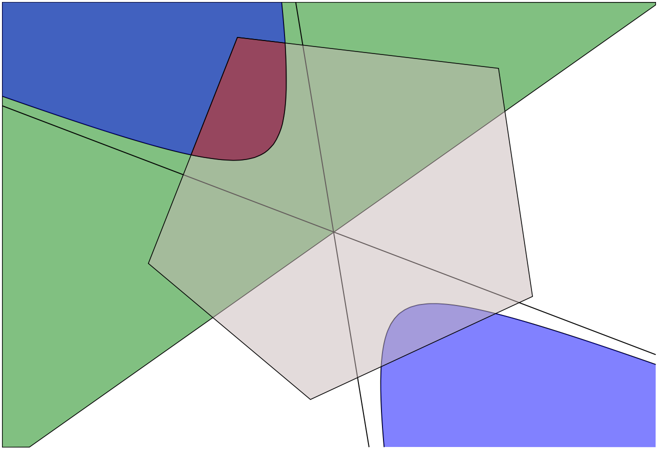
$Y = \{(x, y) \, : \, f(x, y) \leq \gamma\} \cap P^1$



Proof - region P^1 - feasibility problem

Feasibility problem for $Y \cap \mathbb{Z}^2$

$Y = \{(x, y) : f(x, y) \leq \gamma\} \cap P \cap H$



Proof - region P^1

Bounds for binary search

- Upper bound: $\forall (x,y) \in P^1,\ f(x,y) \le \bar{\gamma}$
- Lower bound: We bound separately the monomials in f
  $l_x := \min\{x : (x,y) \in P\},\ u_x := \max\{x : (x,y) \in P\}$
  $l_y := \min\{y : (x,y) \in P\},\ u_y := \max\{y : (x,y) \in P\}$

$$
\begin{aligned}
ax^2 &\ge \min\{0, al_x^2, au_x^2\} \\
cy^2 &\ge \min\{0, cl_y^2, cu_y^2\} \\
bxy &\ge \min\{bl_x l_y, bl_x u_y, bu_x l_y, bu_x u_y\} \\
dx &\ge \min\{dl_x, du_x\} \\
ey &\ge \min\{el_y, eu_y\}
\end{aligned}
$$

Proof - region P^3

Idea

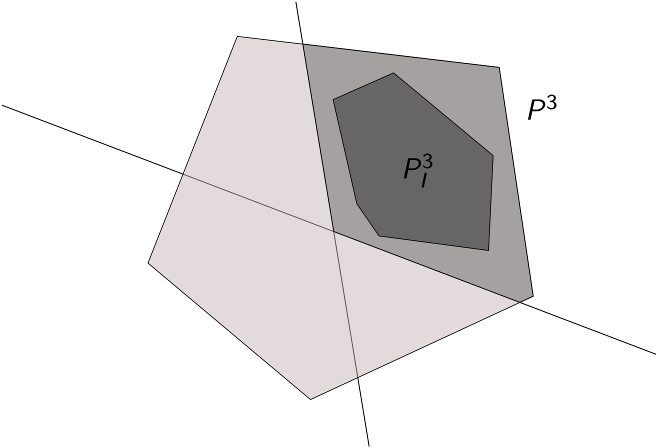
Use Concave Lemma

Concave Lemma

Let $\bar{P}$ be a rational polytope, and let f be quasi-concave on $\bar{P}_I$. Then IQP can be solved in polynomial time.

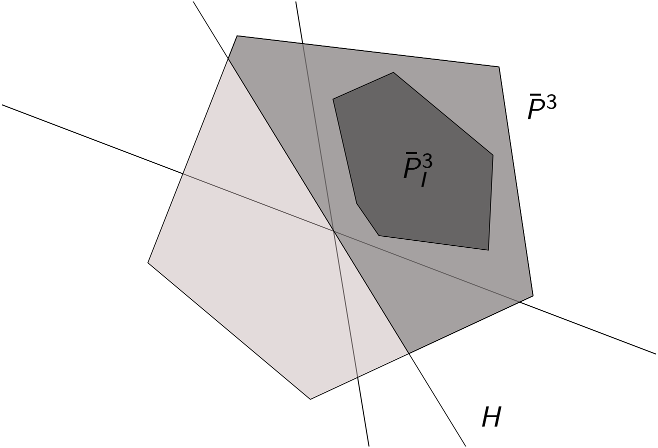
Proof - region P^3

If P^3 is rational we are done



Proof - region P^3

Otherwise we construct a larger rational polyhedron $\bar{P}^3 = P \cap H$



Unbounded polyhedron

Lower bound for binary search for P^1

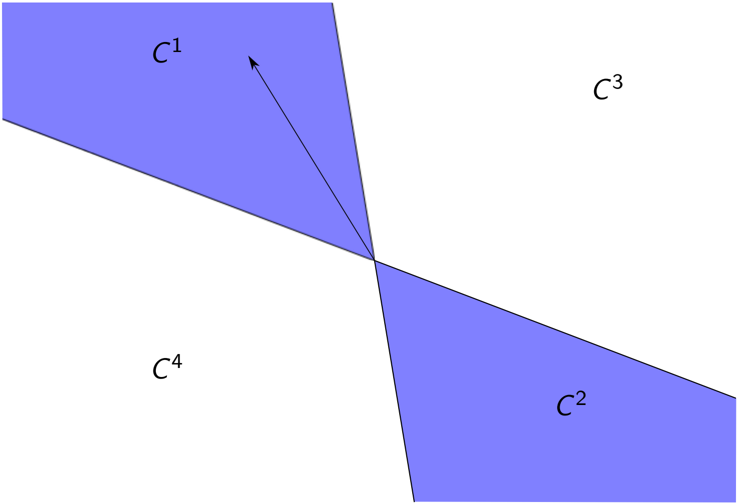
Either P^1 is bounded or the lower bound is $\bar{\gamma}$.

Characterization of boundedness of IQP

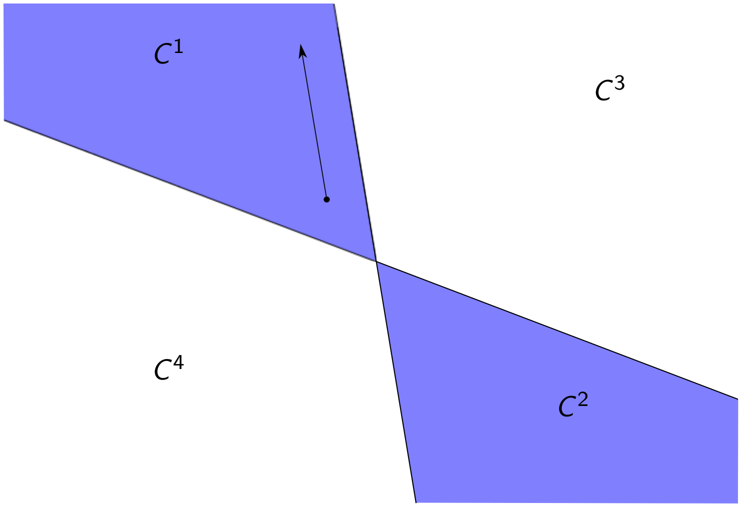
Assume that P contains an integer point.
IQP is unbounded if and only if there exists a nonzero vector $\bar{v}$ in $\mathbb{Z}^2 \cap \operatorname{rec} P$ such that:

- either $a\bar{v}_x^2 + b\bar{v}_x\bar{v}_y + c\bar{v}_y^2 \leq -1$,
- or $a\bar{v}_x^2 + b\bar{v}_x\bar{v}_y + c\bar{v}_y^2 = 0$ and there exists $(\bar{x}, \bar{y}) \in \mathbb{Z}^2 \cap P$ such that $(2a\bar{x} + b\bar{y} + d)\bar{v}_x + (b\bar{x} + 2c\bar{y} + e)\bar{v}_y \leq -1$.

Unbounded polyhedron



Unbounded polyhedron



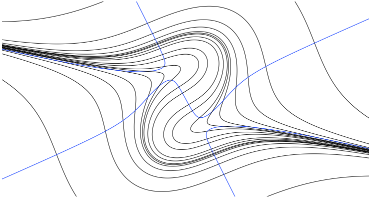
Future directions

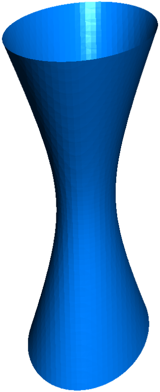
	n=1	n=2	n=3	n fixed	n general
d=1	P	P	P	P	NPH
d=2	P	P	?	?	NPH
d=3	P	?	?	?	NPH
d=4	P	NPH	NPH	NPH	NPH
d general	P	NPH	NPH	NPH	NPH

Future directions

- Quadratics in higher dimension? See Santanu's talk!
- Higher degree in the plane? See Robert H.'s talk!

Future directions





More future directions

- Polynomial constraints
- Multilinear functions
- Cutting planes