Lattice-free sets, multi-branch disjunctions, and mixed-integer programming

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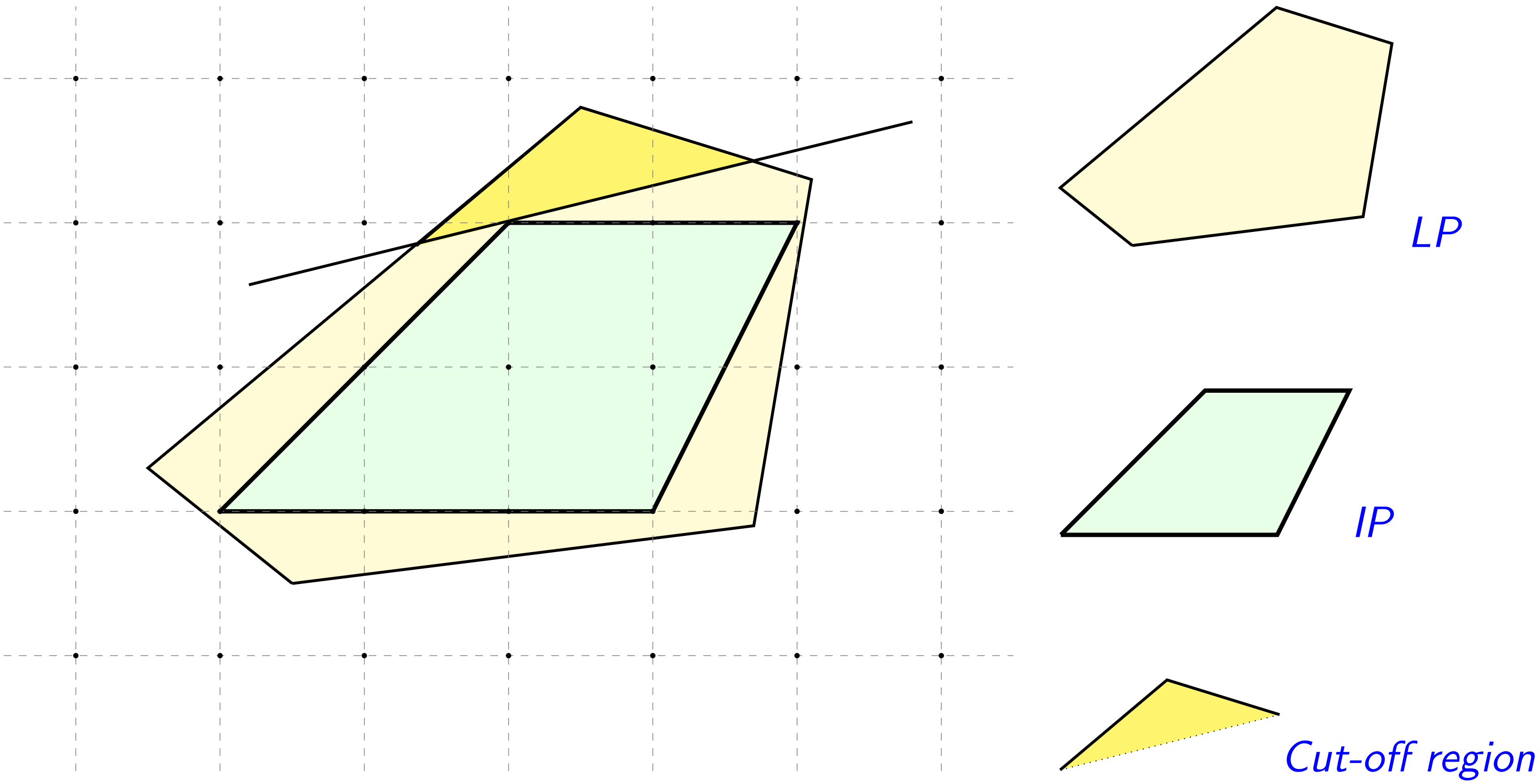
Cutting Planes for MILP



The region cut-off by the valid inequality is:

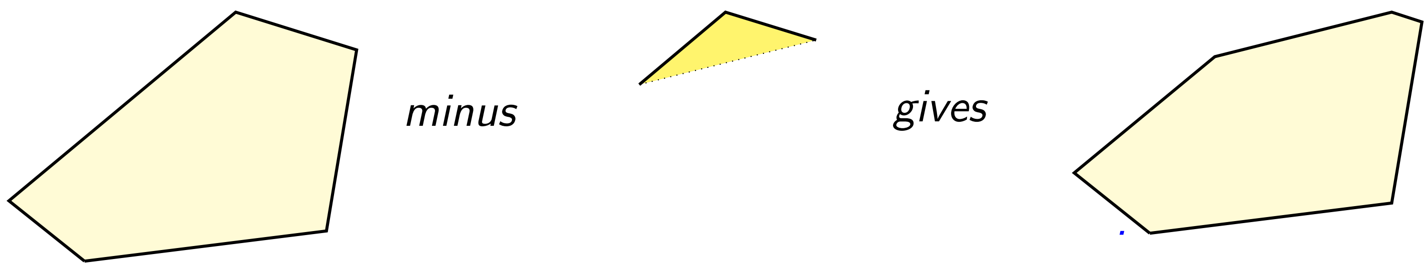
1. *Strictly lattice-free*

2. *Convex*

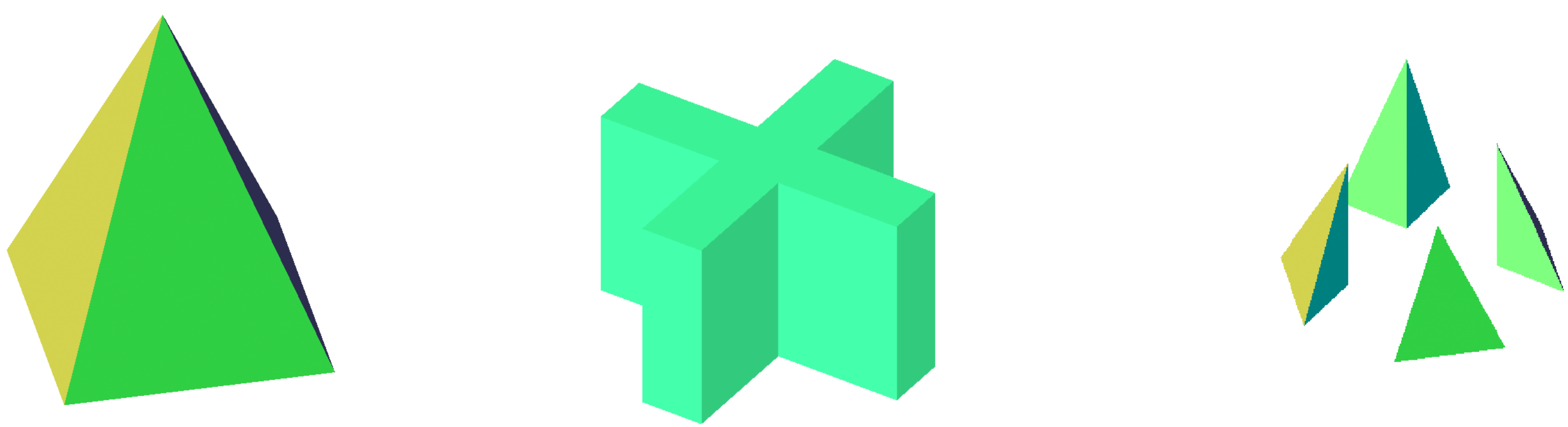
Generating Cutting Planes Using Lattice Free Sets

- *Relaxation minus a strictly lattice-free set gives a tighter relaxation.*

  Ex:



- *But a convexification step might be necessary:*



Outline

We will show that any valid inequality for P is a t-branch split cut for finite t.

- *Chen, Kücükyavuz and Sen (2011) developed the* "cutting tree approach" *to show the same result*
  - *for bounded P, and the number t depends on the data*
- *We remove dependence on data and the boundedness requirement.*
- *Our approach is similar to Lenstra's polynomial time algorithm for MIPs in fixed dimension.*

This also gives a finite cutting-plane algorithm for MIPs

- *Del Pia and Weismantel (2010) show the same result using integral lattice-free cuts.*

How big is the finite t?

- *We construct an example where t grows exponentially with the dimension of P.*

Introduction

Let

$$P = \{(x,v) \in \mathcal{Z}^n \times \mathcal{R}^k \ : \ Ax + Cv \geq d\}$$

where A, C, d *is rational and let* P^{LP} *denote its continuous relaxation.*

Let $D = \cup_{k \in K} D_k$ *where*

$$D_k = \{(x,y) \in \mathcal{R}^{n+l} : \ A^k x \leq b^k\} \qquad \textit{for } k \in K$$

D *is called a* **disjunction** *if* $\mathcal{Z}^n \times \mathcal{R}^l \subseteq D$ *(clearly* $D = D^x \times \mathcal{R}^l$*)*

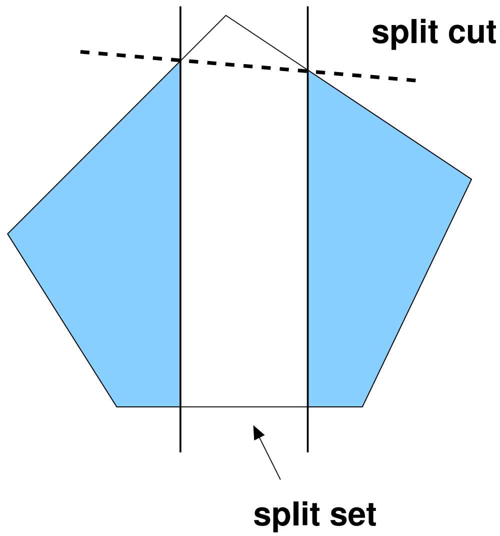
The **disjunctive hull** *of* P *with respect to* D *is*

$$P_D = \mathit{conv}\left(P^{LP} \cap D\right) = \mathit{conv}\Big(\bigcup_{k \in K} (P^{LP} \cap D_k)\Big)$$

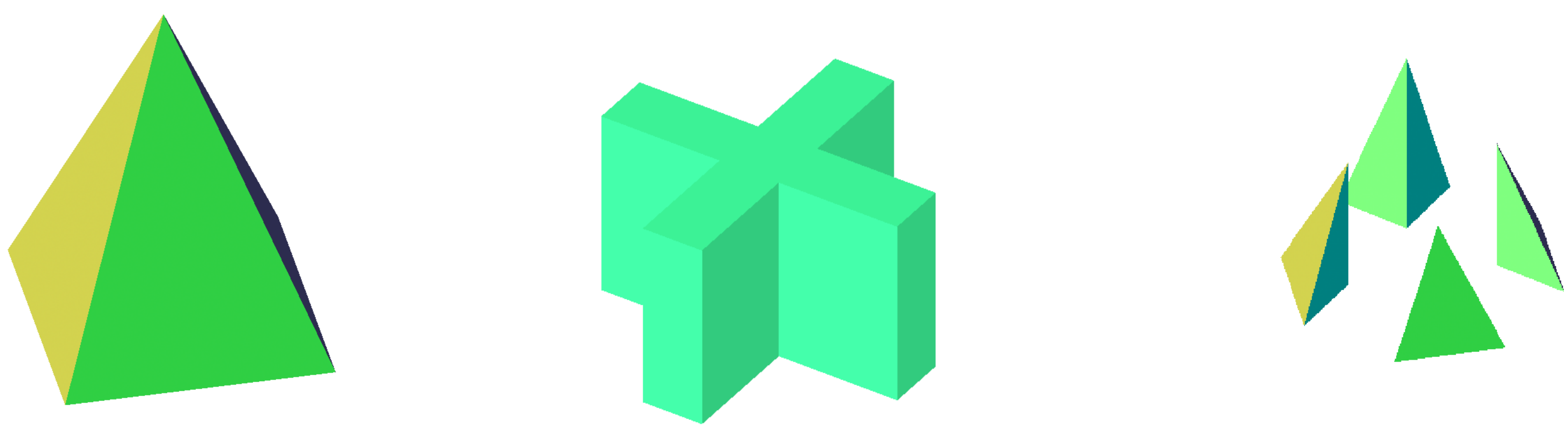
Notice that $P_D = \mathit{conv}\left(P^{LP} \setminus B\right)$ *where* $B = \mathcal{R}^{n+l} \setminus D$ *and it is* **strictly lattice-free**.

Split cuts, cross cuts, ...

- *A* **split cut** *is an inequality valid for* $P^{LP} \setminus S$*:*



- *A* **cross cut** *is an inequality valid for* $P^{LP} \setminus \{S_1 \cup S_2\}$*:*



Valid inequalities as disjunctive cuts

Let $c^T x + d^T y \geq f$ be a valid inequality for P and

$$V = \{(x, y) \in P^{LP} \; : \; c^T x + d^T y < f\}.$$

Clearly $V \cap \mathcal{Z}^n \times \mathcal{R}^l = \emptyset$

Jörg (2007) observes that $V^x \subseteq \text{int}(B)$ where

- $V^x \subset \mathcal{R}^n$ is the projection of V in the space of the integer variables
- B is a polyhedral lattice-free set defined by rational data

$$B = \{x \in \mathcal{R}^n \; : \; \pi_i^T x \geq \gamma_i, \; i \in K\}$$

Therefore $c^T x + d^T y \geq f$ is valid for conv $\left(P^{LP} \setminus \text{int}(\hat{B})\right) \subseteq$ conv $\left(P^{LP} \setminus \hat{V}^x\right)$.

Based on this observation, Jörg then argues that

$$D \;=\; \bigcup_{i \in K} \{(x, y) \in \mathcal{R}^{n+l} \; : \pi_i^T x \leq \gamma_i\}$$

is a valid disjunction and $c^T x + d^T y \geq f$ can be derived from this disjunction.

Valid inequalities as multi-branch split cuts

Let π_i *and* γ_i *be integral for* $i = 1, \ldots, t$ *and consider the split sets*

$$S(\pi_i, \gamma_i) = \{(x, y) \in \mathcal{R}^{n+k} \;:\; \gamma_i < \pi_i^T x < \gamma_i + 1\}$$

A **multi-branch split cut** *is an inequality valid for* $P^{LP} \setminus \bigcup_i S(\pi_i, \gamma_i)$

Remember the points cut off by the valid inequality $c^T x + d^T y \geq f$

$$V = \{(x, y) \in P^{LP} \;:\; c^T x + d^T y < f\}.$$

Fact : *Let* $\mathcal{S} = \cup S_i$ *be a collection of split sets in* $\mathcal{R}^{n+k}$*. If* $V \subseteq \mathcal{S}$*, then* $c^T x + d^T y \geq f$ *is a multi-branch split cut obtained from* $\mathcal{S}$*.*

Question : *Are all facet defining inequalities t-branch split cuts for finite* t*?*
. *(equivalently, can* V *be covered by a finite number of split sets?)*

Lattice width

- *Given a closed, bounded, convex set (or convex* body*)* $B \subseteq \mathcal{R}^n$ *and a vector* $c \in \mathcal{Z}^n$*,*

$$w(B, c) = \max\{c^T x : x \in B\} - \min\{c^T x : x \in B\}.$$

  is the lattice width *of* B *along the direction* c*.*

- *The* lattice width *of* B *is*

$$w(B) = \min_{c \in \mathcal{Z}^n \setminus \{\mathbf{0}\}} w(B, c)$$

  (If the set is not closed, we define its lattice width to be the lattice width of its closure)

- **Khinchine's flatness theorem:** *there exists a function* $f(\cdot) : \mathcal{Z}_+ \to \mathcal{R}_+$ *such that for any strictly lattice-free bounded convex set* $B \subseteq \mathcal{R}^n$*,*

$$w(B) \leq f(n)$$

  where $f(\cdot)$ *depends on the dimension of* B *(not on the complexity of* B*)*

- *Lenstra uses this result to construct a finite enumeration tree to solve the integer feasibility problem.*

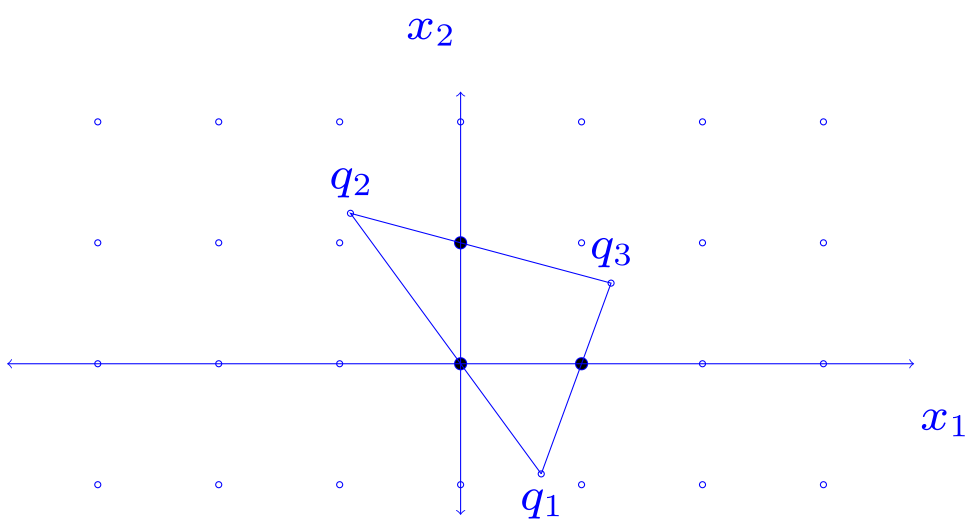
Lattice-free sets in $\mathcal{R}^2$

Theorem : *[Hurkens (1990)] If $B \in \mathcal{R}^2$ is lattice-free, then $w(B) \le 1 + \frac{2}{\sqrt{3}} \approx 2.1547$. Furthermore $w(B) = 1 + \frac{2}{\sqrt{3}}$ if and only if B is a triangle with vertices q_1, q_2, q_3 such that (let $q_4 := q_1$)*

$$\frac{1}{\sqrt{3}}\, q_i + \left(1 - \frac{1}{\sqrt{3}}\right) q_{i+1} = b_i, \text{ for } i = 1, 2, 3.$$

where $b_i \in \mathcal{Z}^2$ for $i = 1, 2, 3$.

The lattice-free triangle T when $b_1 = (0,0)^T$, $b_2 = (0,1)^T$, and $b_3 = (1,0)^T$

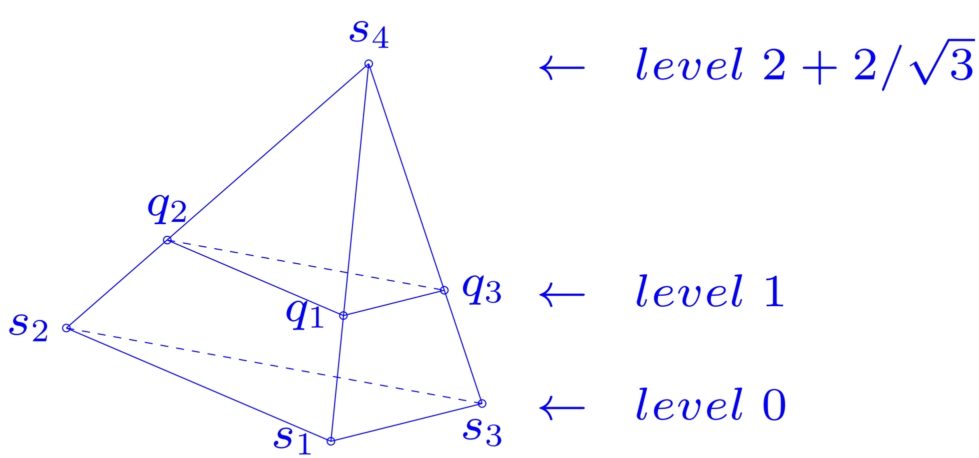


(this is a type 3 triangle)

Lattice-free sets in $\mathcal{R}^3$

Averkov, Wagner and Weismantel (2011) enumerated all maximal lattice-free bodies in $\mathcal{R}^3$ *that are integral. These sets have the lattice width* ≤ 3.

A tetrahedron H *with lattice width* $2 + 2/\sqrt{3} \approx 3.1547$:



where $s_4 = (0, 0, 2+2/\sqrt{3})$, *and* $q_1, \ldots, q_3 \in \mathcal{R}^2$ *are the vertices of Hurken's triangle.*

We can also show that $f(3) \leq 4.25$.

Lattice width of lattice free convex bodies in $\mathcal{R}^n$

- *Given a lattice free convex body* $B \subseteq \mathcal{R}^n$ *the lattice width is*

$$w(B) = \min_{c \in \mathcal{Z}^n \setminus \{\mathbf{0}\}} w(B, c) \leq f(n)$$

- *Lenstra (1983) showed that* $f(n) \leq 2^{n^2}$
- *Kannan and Lovász (1988) showed that* $f(n) \leq c_0(n+1)n/2$ *for some constant* c_0 *(*$c_0 = \max\{1, 4/c_1\}$ *where* c_1 *is another constant defined by Bourgain and Milman)*
- *Banaszczyk, Litvak, Pajor, and Szarek (1999) showed that* $O(n^{3/2})$
- *Rudelson (2000) showed that* $O(n^{4/3} \log^c n)$ *for some constant* c*.*

Valid inequalities as multi-branch split cuts

Let π_i *and* γ_i *be integral for* $i = 1, \ldots, t$ *and consider the split sets*

$$S(\pi_i, \gamma_i) = \{(x, y) \in \mathcal{R}^{n+k} \ : \ \gamma_i < \pi_i^T x < \gamma_i + 1\}$$

A **multi-branch split cut** *is an inequality valid for* $P^{LP} \setminus \bigcup_i S(\pi_i, \gamma_i)$

Remember the points cut off by the valid inequality $c^T x + d^T y \geq f$

$$V = \{(x, y) \in P^{LP} \ : \ c^T x + d^T y < f\}.$$

Fact : *Let* $\mathcal{S} = \cup S_i$ *be a collection of split sets in* $\mathcal{R}^{n+k}$*. If* $V \subseteq \mathcal{S}$*, then* $c^T x + d^T y \geq f$ *is a multi-branch split cut obtained from* $\mathcal{S}$*.*

Question : *Are all facet defining inequalities t-branch split cuts for finite* t*?*
(equivalently, can V *be covered by a finite number of split sets?)*

Bounded case

Lemma : *Let B be a bounded, strictly lattice-free convex set in $\mathcal{R}^n$. Then B is contained in the union of at most $\Pi_{k=1}^n (2 + \lceil f(k) \rceil)$ split sets.*

Proof : *By Khinchine's flatness result.*

- *There is an integer vector $a \in \mathcal{Z}^n$ such that $f(n) \geq u - l$ where*

$$u = \max\{a^T x : x \in B\} \text{ and } l = \min\{a^T x : x \in B\}$$

- *Therefore, $B \subseteq \{x \in \mathcal{R}^n : \lfloor l \rfloor \leq a^T x \leq \lceil u \rceil\}$.*

- *Let U be the collection of the split sets $S(a, b)$ for $b \in V = \{\lfloor l \rfloor, \ldots, \lceil u \rceil - 1\}$*

$$B \setminus \bigcup_{b \in V} S(a, b) = \bigcup_{b \in \bar{V}} \{x \in B : a^T x = b\}$$

  where $\bar{V} = \{\lceil l \rceil, \ldots, \lfloor u \rfloor\}$.

- *All $\{x \in B : a^T x = b\}$ are strictly lattice-free and have dimension at most $n - 1$*

- *Repeating the same argument proves the claim.* ■

Unbounded case

Lemma : *Let B be a strictly lattice-free, convex, unbounded set in $\mathcal{R}^n$ which is contained in the interior of a maximal lattice-free convex set. Then B can be covered by $\Pi_{k=1}^{n}(2 + \lceil f(k) \rceil)$ split sets.*

Proof :

- *Let B' be a maximal lattice free set containing B in its interior.*
- *Lovász (1989) and Basu, Conforti, Cornuejols, Zambelli (2010) showed that*

$$B' = Q + L$$

  where Q is a polytope and L a rational linear space.

- *Let $dim(Q) = d$ and $dim(L) = n - d > 0$.*
- *After a unimodular transformation, $L = \mathcal{R}^{n-d}$*
- *Use the result for the bounded case and the result follows.* ∎

Combining the two cases

Theorem : *Every facet-defining inequality for P is a t-branch split cut for $t = \Pi_{k=1}^{n}(2 + \lceil f(k) \rceil)$.*

- *Let $c^T x + d^T y \geq f$ be valid for conv(P) but not for P^{LP},*
- *Let $V \subseteq \mathcal{R}^{n+l}$ be the set cut off by $c^T x + d^T y \geq f$ and let V^x be its the projection on the space of the integer variables.*
- *V^x is strictly lattice-free, and is non-empty.*
- *Jörg (2007) showed that V^x is contained in the interior of a lattice-free rational polyhedron and therefore in the interior of a maximal lattice-free convex set.*
- *Depending on whether V^x is bounded or unbounded, we can use either of the previous two lemmas to prove the claim.* ■

Note that Jörg's already observed that every facet-defining inequality is a disjunctive cut. We show that these inequalities can be derived as structured (t-branch split) disjunctive cuts.

Theorem : *The mixed-integer program*

$$\min\{c^T x + d^T y \; : \; (x, y) \in \mathcal{Z}^n \times \mathcal{R}^l \, , \; Ax + Gy \geq b\}$$

where the data is rational, can be solved in finite time via a pure cutting-plane algorithm which generates only t-branch split cuts.

Proof : *Let* $t = \Pi_{i=1}^{n}(2 + \lceil f(i) \rceil)$.

- *Represent any t-branch split disjunction $D(\pi_1, \ldots, \pi_t, \gamma_1, \ldots, \gamma_t)$ by $v \in \mathcal{Z}^{(n+1)t}$.*
- *Let $\Omega = \mathcal{Z}^{(n+1)t}$ and arrange its members in a sequence $\{\Omega_i\}$, (by increasing norm)*
- *Let D_i be the t-branch split disjunction defined by Ω_i.*
- *Any facet-defining inequality of conv (P), is a t-branch split cut defined by the disjunction D_k for some (finite) k.*
- *Let k^* be the largest index of a disjunction associated with facet-defining inequalities.*
- *Solve the relaxation of the MIP for $P_i = P_{i-1} \cap \text{conv}\,(P_0 \cap D_i)$. for $i = 1, 2, \ldots$* ■

Note: Validity of a given inequality can also be checked by changing the termination criterion. In addition conv (P), can also be computed the same way.

How finite is this algorithm?

Theorem : *The mixed-integer program*

$$\min\{c^T x + d^T y \ : \ (x, y) \in \mathcal{Z}^n \times \mathcal{R}^l \, , \ Ax + Gy \geq b\}$$

where the data is rational, can be solved in finite time via a pure cutting-plane algorithm which generates only t-branch split cuts.

Proof : *The algorithm cannot run forever* ■

Stronger Result: *The runtime of this algorithm is bounded.*

Proof : *The LP relaxation P^{LP} has bounded complexity (number of bits to represent facets defining inequalities)*

$\Rightarrow$ *Therefore conv (P) has bdd complexity.*

$\Rightarrow$ *Therefore the set of points cut-off by a facet has bdd complexity.*

$\Rightarrow$ *Therefore the multi-branch disjunction needed to generate a facet has bdd complexity.*

$\Rightarrow$ *Order the disjunctions in increasing complexity.* ■

Part II

How finite is t?

- *We showed that every facet-defining inequality for P is a t-branch split cut for $t = \Pi_{k=1}^{n}(2 + \lceil f(k) \rceil)$.*

  [best know bound $f(k) \leq O(k^{4/3} \log^c k)$ by Rudelson, 2000].

- *It is easy to show that $t \geq \Omega(n)$*

- *We next show that $t \geq \Omega(2^n)$*

An exponential bound on t

Theorem : *For any $n \geq 3$ there exists a nonempty rational mixed-integer polyhedral set in $\mathcal{Z}^n \times \mathcal{R}$ with a facet-defining inequality that cannot be expressed as a $3 \times 2^{n-2}$-branch split cut.*

.

Proof : .

- *Construct a full-dimensional rational, lattice-free polytope $B \subset \mathcal{R}^n$ such that*
  - *Its interior cannot be covered by $3 \times 2^{n-2}$ split sets*
  - *The integer hull of $B \subset \mathcal{R}^n$ has dimension n*
- *Define a mixed-integer polyhedral set P_B as follows:*

$$P_B = \{(x, y) \in \mathcal{Z}^n \times \mathcal{R} : (x, y) \in B'\}.$$

where

$$B' = \text{conv}\,((B \times \{-1\}) \cup (B \times \{0\}) \cup (\bar{x} \times \{1/2\}))$$

and $\bar{x}$ is a point in the interior of B.

- *$y \leq 0$ is a facet-defining inequality for* $\text{conv}\,(P_B)$
- *To cover*

$$V = \{(x, y) \in P_B^{LP} \ : \ y > 0\}$$

one needs at least $(3 \times 2^{n-2})$ split sets. ■

How to construct the lattice-free polytope $B \subset \mathcal{R}^n$

For $\Delta \in \{0, \ldots, 2^{n-2}-1\}$, *let* $T_\Delta \in \mathcal{R}^2$ *be a (rational) lattice-free triangle and* $\mathrm{cent}(T_\Delta)$ *denote its* centroid,

Let $\Delta = \sum_{l=1}^{n-2} \delta_l 2^{l-1}$ *with* $\delta_l \in \{0, 1\}$

$$\boxed{B := \mathit{conv}\Big(\bigcup_{\Delta=0}^{2^{n-2}-1} (\mathbf{T}_\Delta \cup \{p_{\varepsilon,\Delta}\})\Big)}$$

where

$$\mathbf{T}_\Delta := \{(\delta_1, \ldots, \delta_{n-2}, x, y) | (x, y) \in T_\Delta\}$$

and

$$p_{\varepsilon,\Delta} := (\delta_1, \ldots, \delta_{n-2}, \mathrm{cent}(T_\Delta)) + ((2\delta_1 - 1)\varepsilon, \ldots, (2\delta_{n-2} - 1)\varepsilon, 0, 0)$$

(For example, $p_{\varepsilon,0} = (-\varepsilon, \ldots, -\varepsilon, \bar{x}, \bar{y})$ *where* $(\bar{x}, \bar{y}) = \mathrm{cent}(T_0)$*.)*

B *has the following properties: (i) it is rational, (ii) it is lattice-free, (iii) it has a full-dimensional integer hull (iv) it contains* $\mathbf{T}_\Delta$ *in its interior for all* Δ*.*

(to cover the interior of B*, you need to cover all* $\mathbf{T}_\Delta$*)*

How to construct the triangles $\mathrm{T}_\triangle$

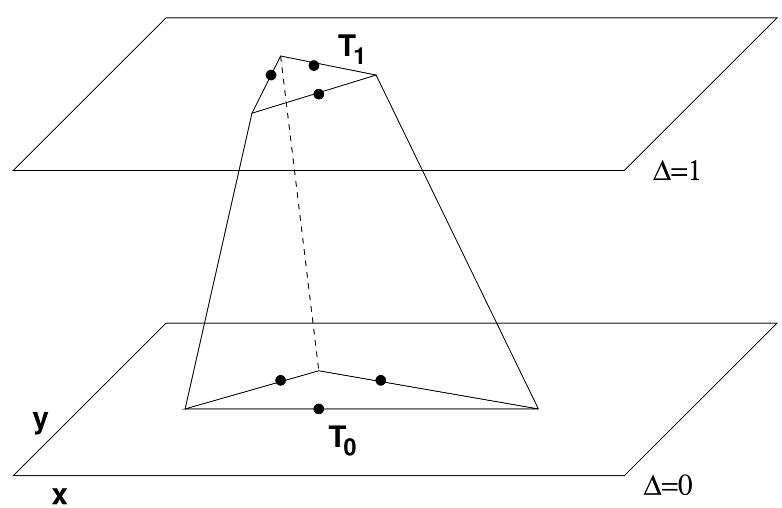
- $\mathbf{T}_0 \in \mathcal{R}^2$ *is a rational Hurken's triangle with* $w(\mathbf{T}_0) \geq 2.15$ *that needs at least 3 split sets to cover.*
- *For* $\Delta \in \{1, \ldots, 2^{n-2} - 1\}$,

$$T_\triangle = M_\triangle \mathbf{T}_0$$

where $M_\triangle$ is a 2x2 unimodular matrix with the property that:

  * If a split set is useful in covering some $T_\triangle$, it is not useful for $T'_\triangle$ unless $\Delta = \Delta'$

when $n = 3$



How to construct the unimodular matrices M_Δ

1. **Useful split sets are finite**

   For any compact set $K \subset \mathcal{R}^n$ *and any number* $\varepsilon > 0$, *the collection of split sets* $S(a,b)$ *such that* $vol(K \cap S(a,b)) \geq \varepsilon$ *is finite.*

2. **Useful sets are really necessary**

   For any fixed $l \geq 0$, *there exists a finite set* $\Sigma \subset \mathcal{Z}^2$ *such that if a collection of* l *split sets cover* $\mathbf{T}_0$, *then at least* 3 *of them are contained in* Σ.

3. **Bending the triangles**

   Given any two finite sets of vectors $V, W \subseteq \mathcal{Z}^2 \setminus \{\mathbf{0}\}$, *there exists an unimodular matrix* M *such that* $MV \subseteq \mathcal{Z}^2 \setminus \{\mathbf{0}\}$ *and* $MV \cap W = \emptyset$.

   Proof : *Let* $q = \max_{v \in W} ||v||_\infty$ *then*

$$M = \begin{pmatrix} 1 & \mu \\ \mu & \mu^2 + 1 \end{pmatrix} \text{ where } \mu = 3q$$

■

thank you...