Constraint Programming Techniques in MINLP

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What can CP contribute to MINLP?

- **Interval propagation** (range reduction).
  - Already used in global optimization solvers.
- Focus on lesser-known techniques:
  - **Domain filtering** with Lagrange multipliers.
  - Efficient representation of **piecewise linear** functions.
  - Branching on **multiple discrete values**.
  - Bounds from **quasi-relaxations**.
  - Optimization/propagation in **decision diagrams**.
  - Management of **McCormick factors** with global constraints and semantic typing (*no time for this today*).

Constraint Programming Perspective

- All (successful) optimization method combine **search** with **relaxation** and **inference**.
- Math programming focuses on **relaxation**.
    - LP, Lagrangean, etc.
- Constraint programming (CP) focuses on **inference**.
    - Domain filtering, constraint propagation

Constraint Programming Perspective

- All (successful) optimization method combine **search** with **relaxation** and **inference**.
- Math programming focuses on **relaxation**.
    - LP, Lagrangean, etc.
- Constraint programming (CP) focuses on **inference**.
    - Domain filtering, constraint propagation
- Math programming uses inference…
    - To generate cutting planes, Benders cuts, etc.
    - But the purpose is to strengthen the **relaxation**.

Constraint Programming Perspective

- CP uses inference for **consistency maintenance**
  - …rather than to strengthen a relaxation.
- Greater consistency means **less backtracking** during search.
  - The concept of consistency never developed in math programming, but it helps to explain search behavior.

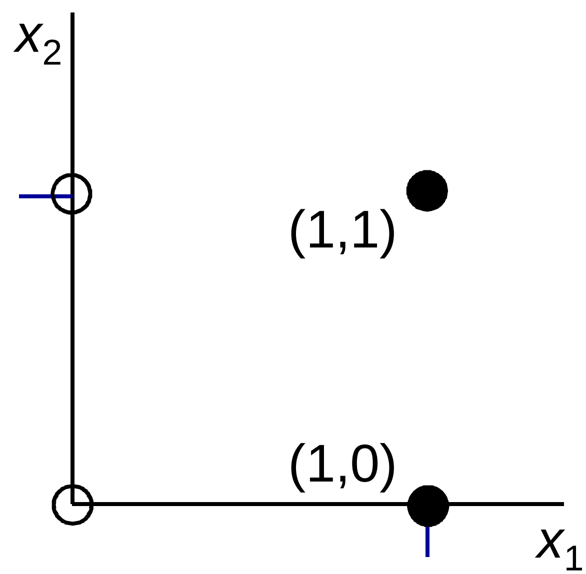
Constraint Programming Perspective

- CP uses inference for **consistency maintenance**
  - …rather than to strengthen a relaxation.
- Greater consistency means **less backtracking** during search.
  - The concept of consistency never developed in math programming, but it helps to explain search behavior.
- Several types of consistency
  - Domain consistency (generalized arc consistency)
  - Bounds consistency
  - Strong k-consistency
  - Etc. etc.

Domain consistency

A constraint set is **domain consistent** if the domain of each variable x_i is the projection of the feasible set onto x_i.

$$
\begin{aligned}
x_1 + x_2 &\geq 1 \\
x_1 - x_2 &\geq 0 \\
x_1, x_2 &\in \{0,1\}
\end{aligned}
$$



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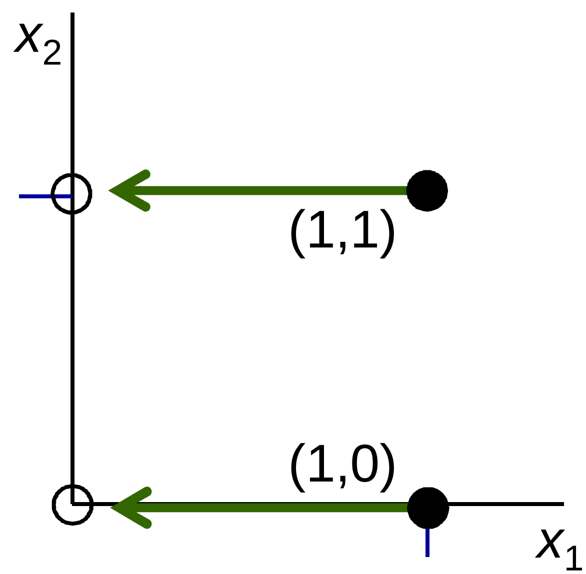

Projection onto $x_2 = \{0,1\}$

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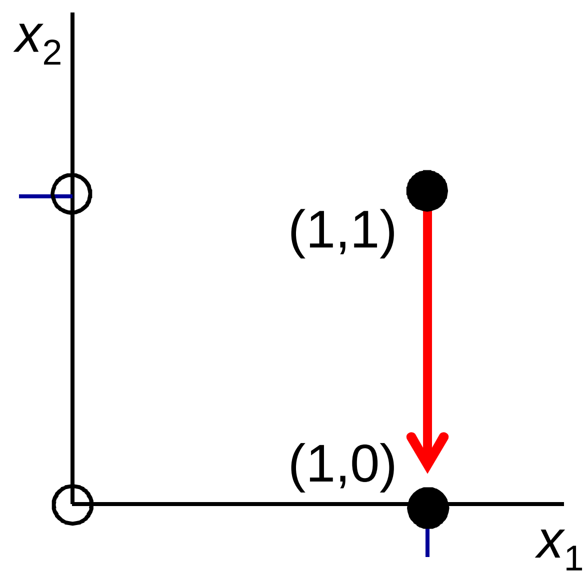


Not domain consistent.

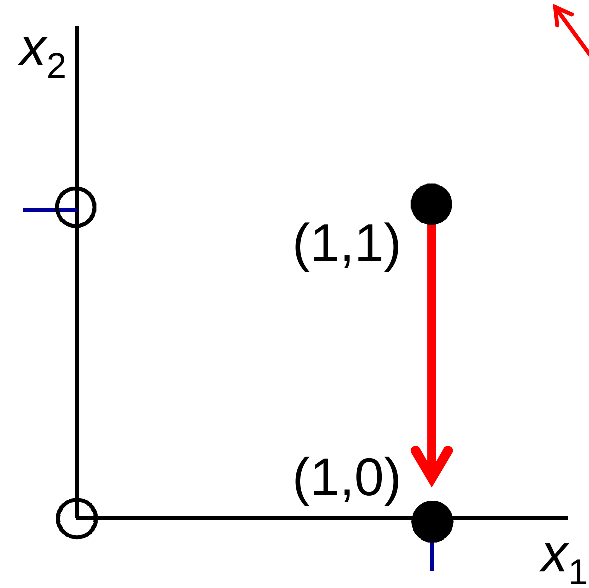
Projection onto $x_1 = \{1\}$

Domain consistency

A constraint set is **domain consistent** if the domain of each variable x_i is the projection of the feasible set onto x_i.

$$x_1 + x_2 \geq 1$$
$$x_1 - x_2 \geq 0$$
$$x_1 \in \{1\}, x_2 \in \{0,1\}$$

Achieve domain consistency by **filtering** 0 from domain of x_1.



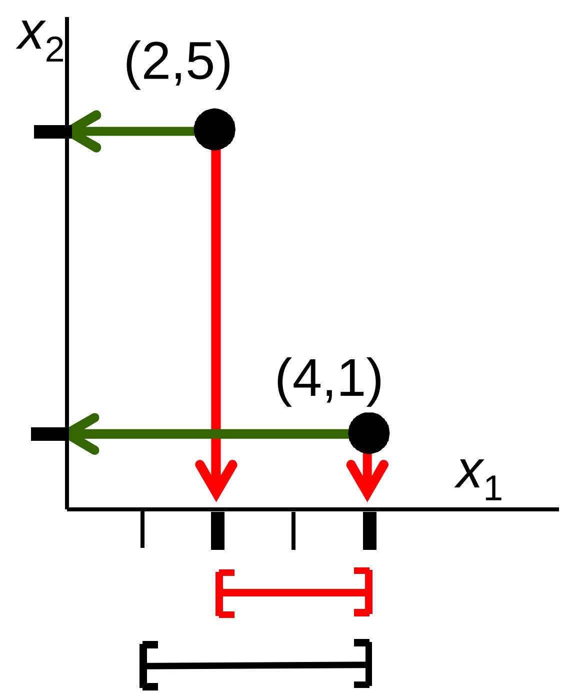
Projection onto $x_2 = \{0,1\}$

Projection onto $x_1 = \{1\}$

Bounds consistency

- A constraint set is **bounds consistent** if the domain of each variable x_i spans the **same interval** as the projection of the feasible set onto x_i.

$$2x_1 + x_2 = 9$$
$$x_1 \in \{1, 2, 3, 4\}$$
$$x_2 \in \{1, 5\}$$



Not bounds consistent.

Consistency maintenance

- Domain or bounds consistency is normally achieved (if at all) for **one constraint at a time**.
    - This can be NP-hard.
    - But allows one to exploit **special structure** of constraints.
    - Much as cutting planes exploit structure of certain classes of inequalities.

Consistency maintenance

- Domain or bounds consistency is normally achieved (if at all) for **one constraint at a time**.
  - This can be NP-hard.
  - But allows one to exploit **special structure** of constraints.
  - Much as cutting planes exploit structure of certain classes of inequalities.
- Particularly effective when the model consists of **global constraints**.
  - …which represent a set of simpler constraints.
  - Alldiff, cardinality, element, nvalues, sequence, circuit, path, regular, cumulative, stretch, etc.

Propagation

- Reduced domains are passed (**propagated**) to the next constraint.
    - …which may allow further reduction.
    - Generally does not achieve consistency for entire constraint set.
    - But it drastically reduces backtracking.

Bounds propagation

- Bounds obtained by achieving bound consistency can be propagated.
    - This is important in global optimization (**range reduction**).

x_1

Bounds propagation

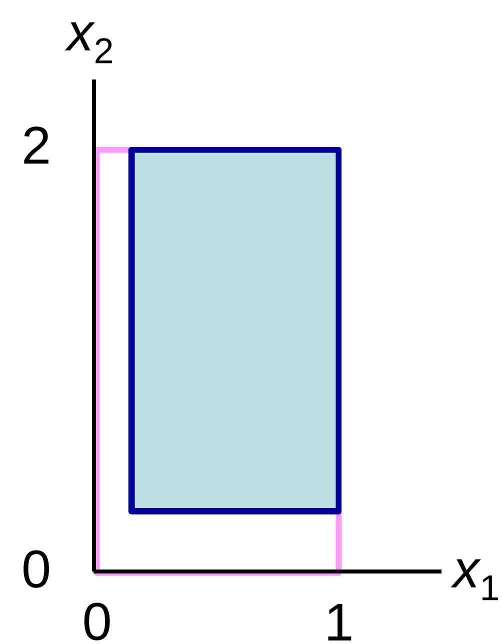
- Bounds obtained by achieving bound consistency can be propagated.
  - This is important in global optimization (**range reduction**).
- Example:

$$
\begin{aligned}
&4x_1x_2 = 1 \\
&2x_1 + x_2 \le 2 \\
&x_1 \in [0.125, 1] \\
&x_2 \in [0.25, 2]
\end{aligned}
$$

Filter using constraint 1:

$$x_1 = \frac{1}{4x_2} \ge \frac{1}{4 \cdot 2} = 0.125$$

$$x_2 = \frac{1}{4x_1} \ge \frac{1}{4 \cdot 1} = 0.25$$



Bounds propagation

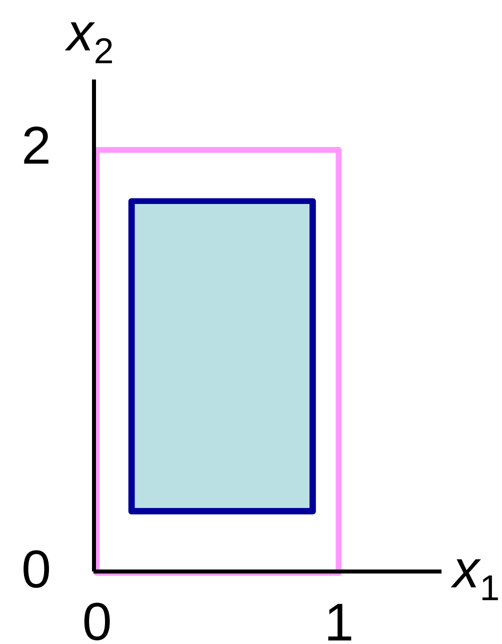
- Bounds obtained by achieving bound consistency can be propagated.
  - This is important in global optimization (**range reduction**).
- Example:

$$4x_1x_2 = 1$$
$$2x_1 + x_2 \le 2$$
$$x_1 \in [0.125, 0.875]$$
$$x_2 \in [0.25, 1.75]$$

Propagate to constraint 2:

$$x_1 \le 1 - \frac{x_2}{2} \le \frac{0.25}{2} = 0.875$$

$$x_2 \le 2 - 2x_1 \le 2 - 2 \cdot 0.125 = 1.75$$



Bounds propagation

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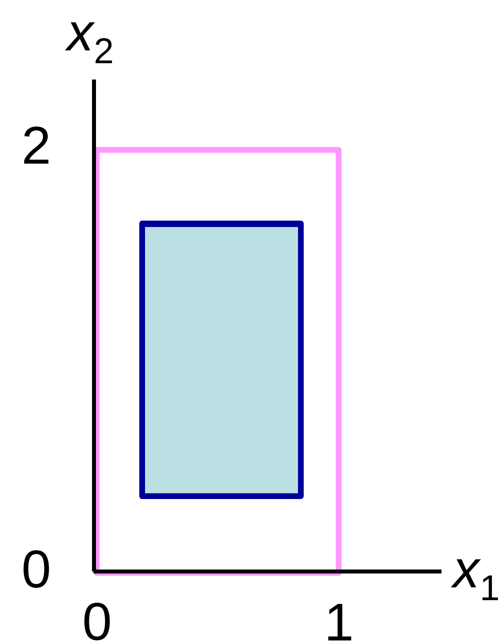
  - This is important in global optimization (**range reduction**).
- Example:

$$4x_1x_2 = 1$$
$$2x_1 + x_2 \le 2$$
$$x_1 \in [0.146, 0.854]$$
$$x_2 \in [0.293, 1.707]$$

Continuing, bounds asymptotically converge:



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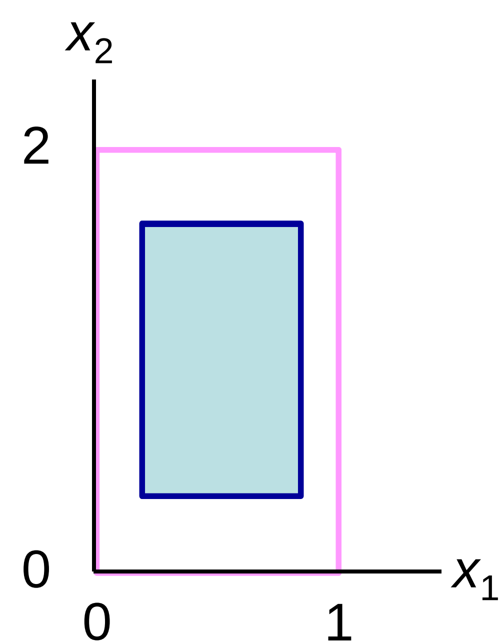


Continuing, bounds asymptotically converge:

Solvers truncate the process.

k-consistency

- *k*-consistency is closely related to backtracking and the **dependency graph** of a constraint set.
    - A constraint set is ***k*-consistent** if any assignment to $k - 1$ variables that violates no constraints can be extended to an assignment to k variables without violating any constraints.

x_{j_k}

k-consistency

- Example

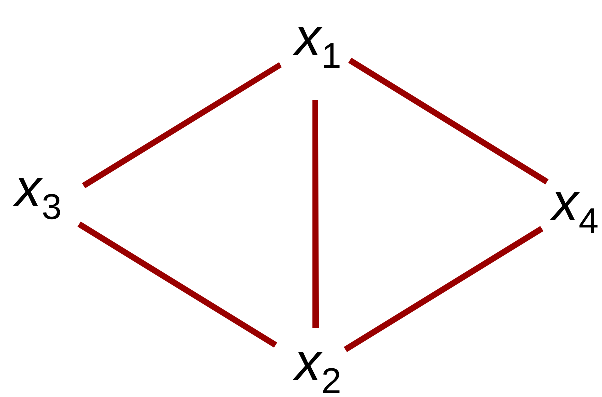
$$\begin{aligned} x_1 + x_2 \qquad\quad + x_4 &\ge 1 \\ x_1 - x_2 + x_3 \qquad &\ge 0 \\ x_1 \qquad\qquad\quad - x_4 &\ge 0 \\ x_j \in \{0,1\} & \end{aligned}$$

  - 2-consistent.
  - not 3-consistent:
    $(x_1,x_2) = (0,0)$ cannot be extended to $(x_1,x_2,x_4) = (0,0,?)$.

Dependency graph

- **Dependency graph**: variables are connected by edges when they occur in a common constraint.

$$\begin{aligned} x_1 + x_2 \qquad + x_4 &\geq 1 \\ x_1 - x_2 + x_3 \qquad &\geq 0 \\ x_1 \qquad\qquad - x_4 &\geq 0 \\ x_j \in \{0,1\} \end{aligned}$$

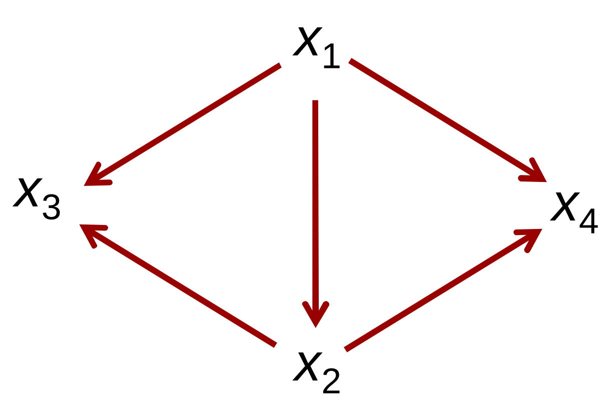


Dependency graph for the example.

Dependency graph

- **Dependency graph**: variables are connected by edges when they occur in a common constraint.

$$\begin{aligned} x_1 + x_2 \phantom{{}+x_3} + x_4 &\ge 1 \\ x_1 - x_2 + x_3 \phantom{{}+x_4} &\ge 0 \\ x_1 \phantom{{}+x_2+x_3} - x_4 &\ge 0 \\ x_j \in \{0,1\} & \end{aligned}$$



For a given variable ordering, **width** of the graph is the maximum in-degree

Here, width = 2
for ordering 1,2,3,4

Backtracking

- A constraint set is **strongly** k-consistent if it is i-consistent for $i = 1,\ldots,k$.

Theorem (Freuder). If the dependency graph has width $< k$ for some variable ordering, then branching (in that order) solves a strongly k-consistent problem **without backtracking**.

Backtracking

- The example doesn't satisfy the conditions of the theorem.
  - Width = 2, not strongly 3-consistent.
  - Backtracking occurs when we set

  $(x_1,x_2,x_3,x_4) = (0,0,0,?)$

$$
\begin{aligned}
x_1 + x_2 \phantom{{}+x_3} + x_4 &\geq 1 \\
x_1 - x_2 + x_3 \phantom{{}+x_4} &\geq 0 \\
x_1 \phantom{{}+x_2+x_3} - x_4 &\geq 0 \\
x_j \in \{0,1\}
\end{aligned}
$$

Backtracking

- Suppose we add two constraints:
  - This is strongly 3-consistent.
  - Backtracking does not occur.

$$
\begin{aligned}
x_1 + x_2 \qquad\quad + x_4 &\geq 1 \\
x_1 - x_2 + x_3 \qquad\quad &\geq 0 \\
x_1 \qquad\qquad\quad - x_4 &\geq 0 \\
x_1 + x_2 \qquad\qquad\quad &\geq 1 \\
x_1 \qquad + x_3 \qquad\quad &\geq 1 \\
x_j \in \{0,1\} &
\end{aligned}
$$

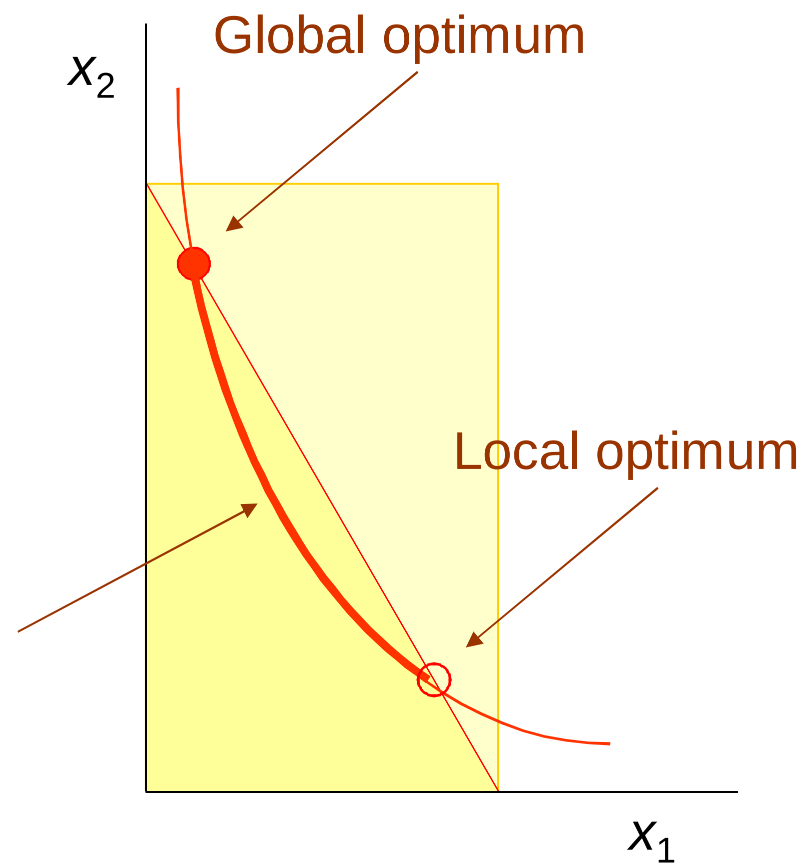
Backtracking

- Suppose we add two constraints:
  - This is strongly 3-consistent.
  - Backtracking does not occur.
- These are valid cuts!
  - Cuts reduce backtracking by increasing the degree of consistency as well as by strengthening the LP relaxation.

$$
\begin{array}{llllll}
x_1 & + x_2 & & + x_4 & \geq 1 \\
x_1 & - x_2 & + x_3 & & \geq 0 \\
x_1 & & & - x_4 & \geq 0 \\
x_1 & + x_2 & & & \geq 1 \\
x_1 & & + x_3 & & \geq 1 \\
\multicolumn{5}{l}{x_j \in \{0,1\}}
\end{array}
$$

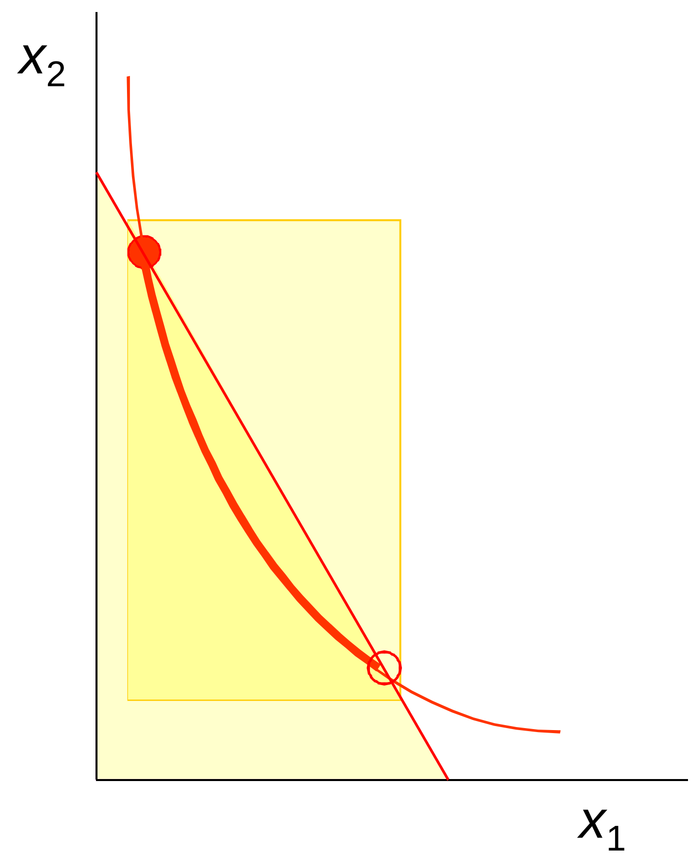
Global optimization example

$$\max \ x_1 + x_2$$
$$4x_1x_2 = 1$$
$$2x_1 + x_2 \le 2$$
$$x_1 \in [0,1], \ \ x_2 \in [0,2]$$



Interval propagation (range reduction)

Propagate intervals
[0,1], [0,2]
through constraints
to obtain
[1/8,7/8], [1/4,7/4]



Relaxation (function factorization)

Factor complex functions into elementary functions that have known linear relaxations (**McCormick factors**).

Write $4x_1x_2 = 1$ as $4y = 1$ where $y = x_1x_2$.

This factors $4x_1x_2$ into linear function $4y$ and bilinear function x_1x_2.

Linear function $4y$ is its own linear relaxation.

Relaxation (function factorization)

Factor complex functions into elementary functions that have known linear relaxations (**McCormick factors**).

Write $4x_1x_2 = 1$ as $4y = 1$ where $y = x_1x_2$.

This factors $4x_1x_2$ into linear function $4y$ and bilinear function x_1x_2.

Linear function $4y$ is its own linear relaxation.

Bilinear function $y = x_1x_2$ has relaxation:

$$\underline{x}_2 x_1 + \underline{x}_1 x_2 - \underline{x}_1 \underline{x}_2 \le y \le \underline{x}_2 x_1 + \bar{x}_1 x_2 - \bar{x}_1 \underline{x}_2$$
$$\bar{x}_2 x_1 + \bar{x}_1 x_2 - \bar{x}_1 \bar{x}_2 \le y \le \bar{x}_2 x_1 + \underline{x}_1 x_2 - \underline{x}_1 \bar{x}_2$$

where domain of x_j is $[\underline{x}_j, \bar{x}_j]$

Relaxation (function factorization)

The linear relaxation becomes:

$$\begin{aligned}
&\min \;\; x_1 + x_2 \\
&4y = 1 \\
&2x_1 + x_2 \le 2 \\
&\underline{x}_2 x_1 + \underline{x}_1 x_2 - \underline{x}_1 \underline{x}_2 \le y \le \underline{x}_2 x_1 + \overline{x}_1 x_2 - \overline{x}_1 \underline{x}_2 \\
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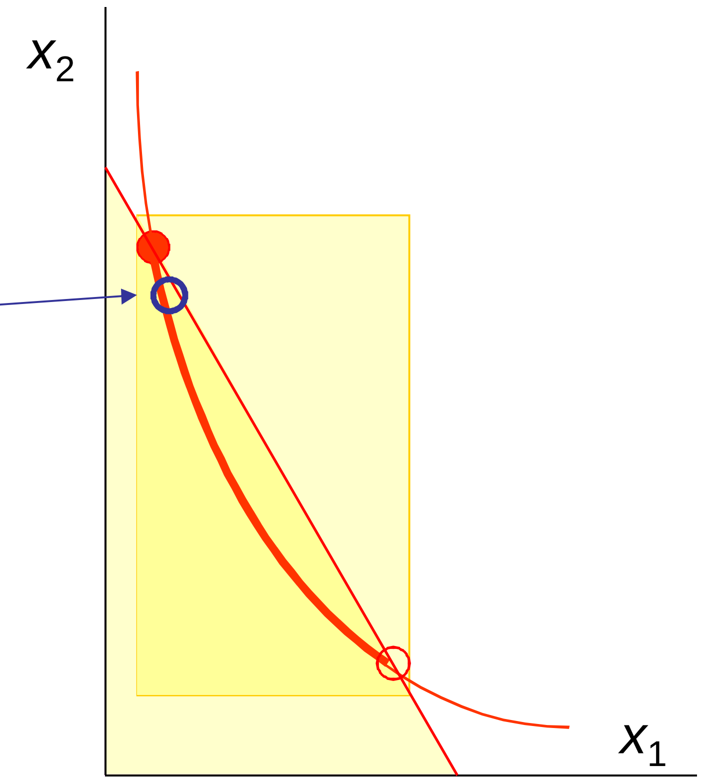
&\underline{x}_j \le x_j \le \overline{x}_j, \;\; j = 1,2
\end{aligned}$$

Relaxation (function factorization)

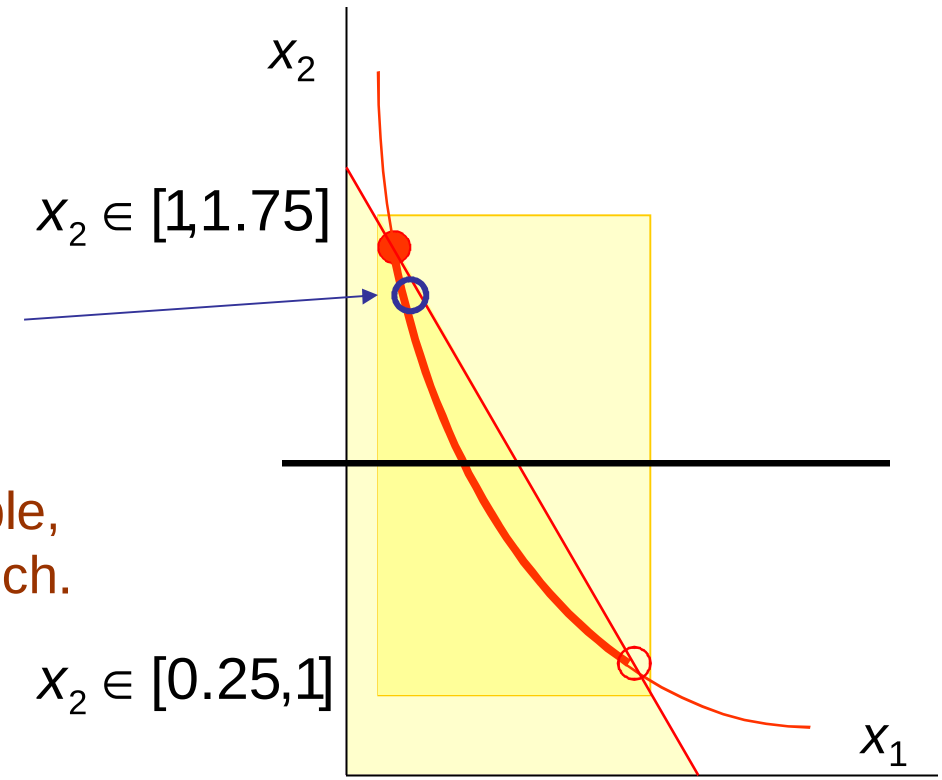
Solve linear relaxation.

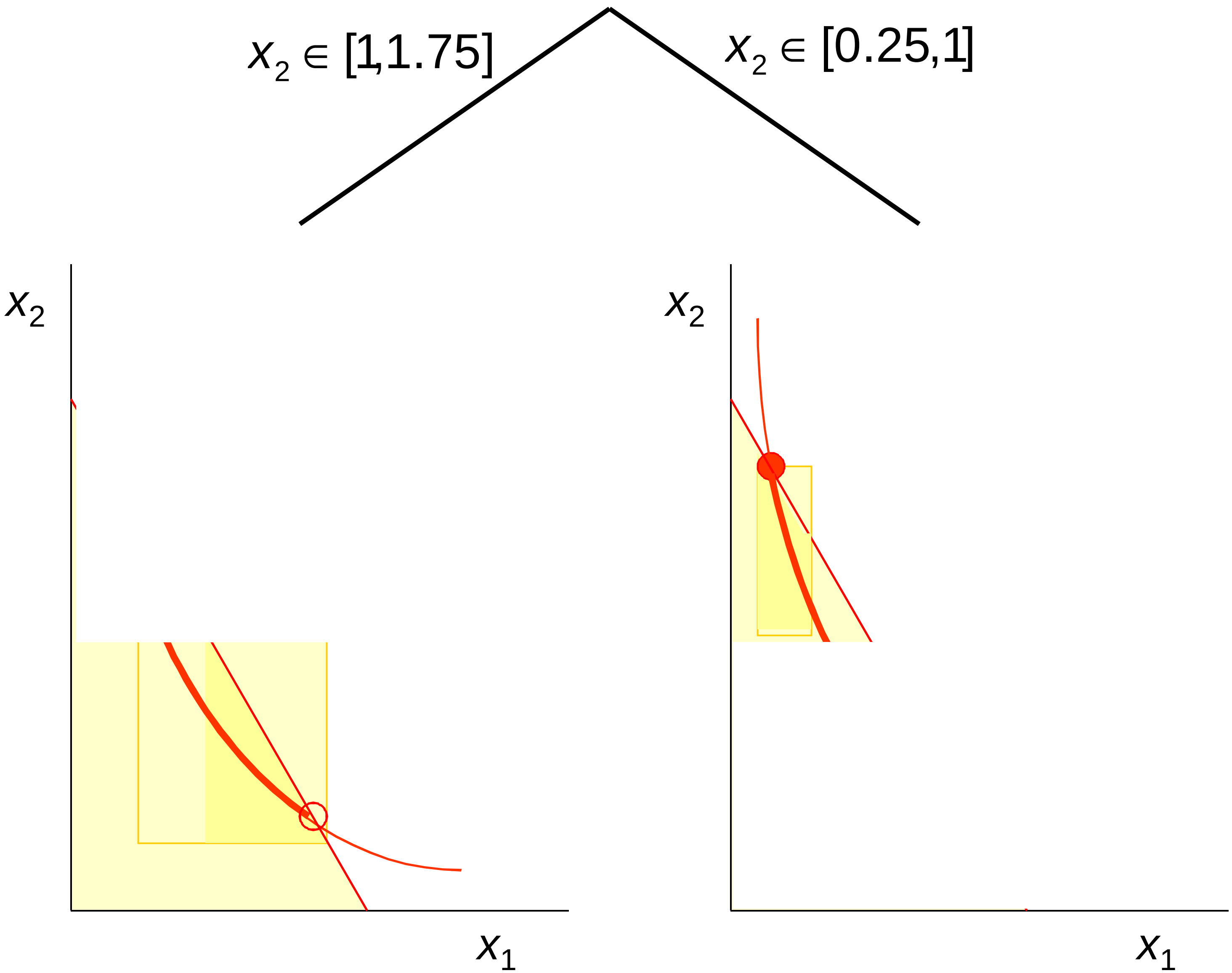


Relaxation (function factorization)

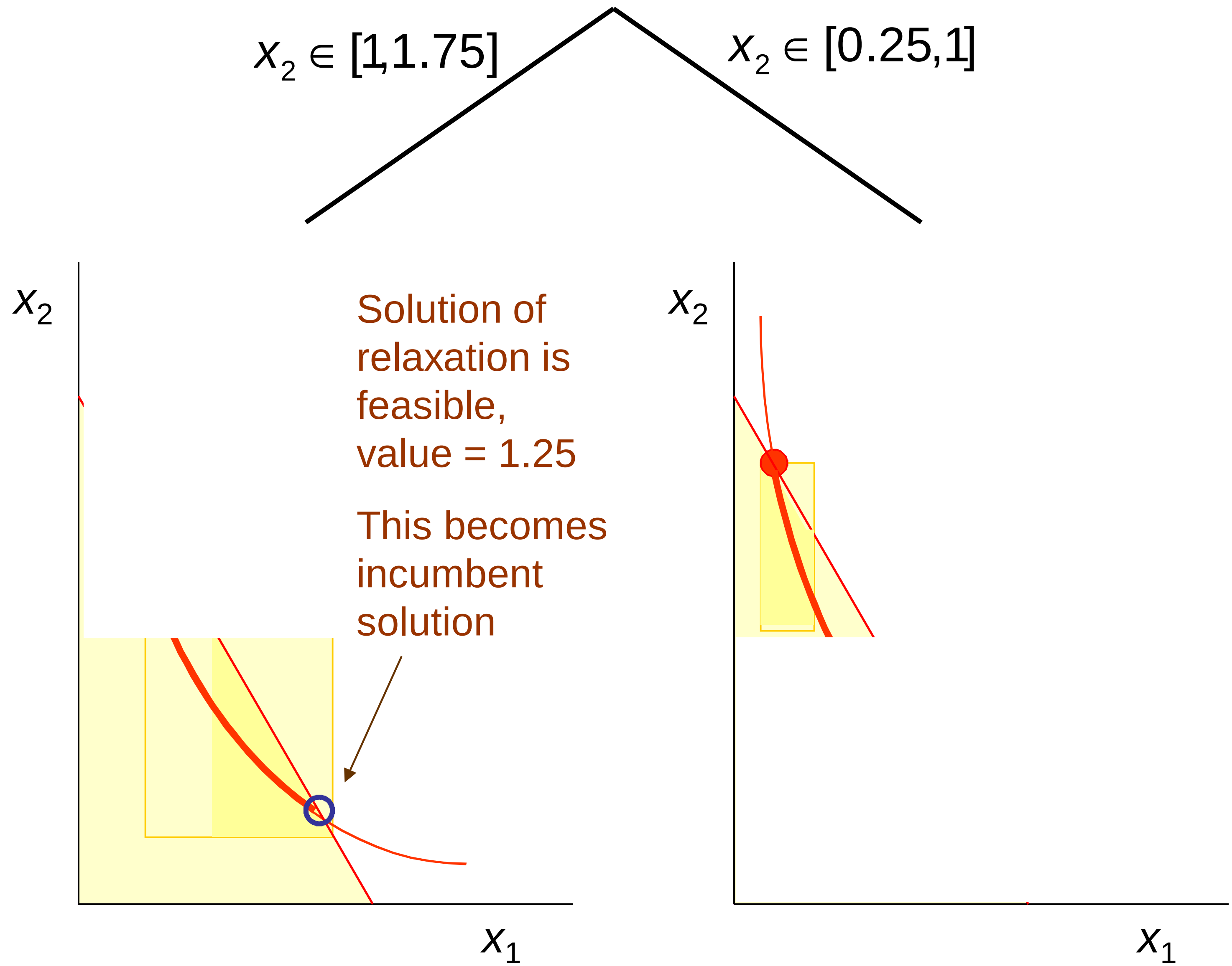
Solve linear relaxation.

Since solution is infeasible, split an interval and branch.

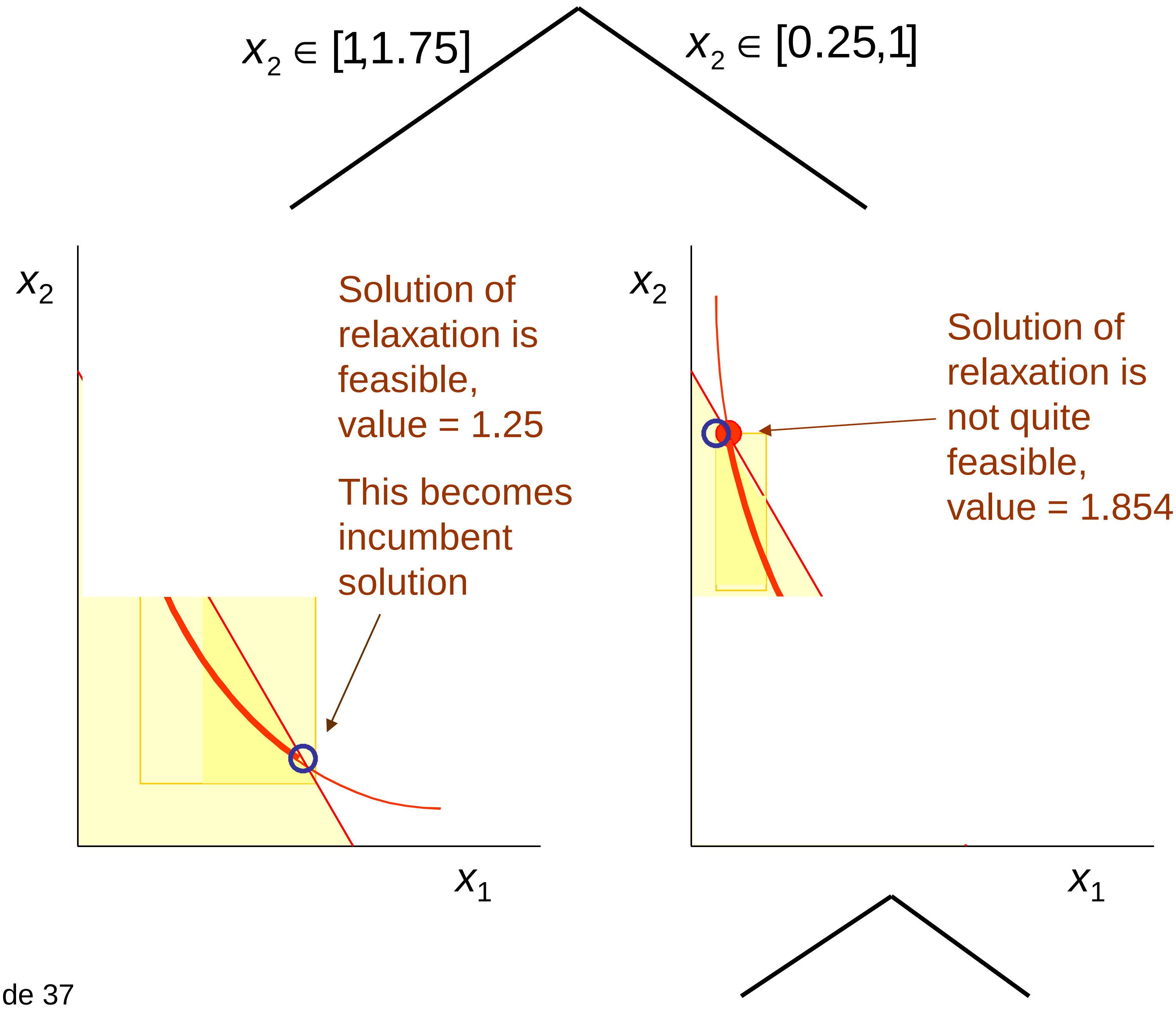




$x_2 \in [1,1.75]$
$x_2 \in [0.25,1]$
x_2
x_1
x_2
x_1



$x_2 \in [1,1.75]$
$x_2 \in [0.25,1]$
x_2
x_1
Solution of relaxation is feasible, value = 1.25
This becomes incumbent solution
x_2
x_1


$x_2 \in [1,1.75]$
$x_2 \in [0.25,1]$
x_2
x_1
Solution of relaxation is feasible, value = 1.25
This becomes incumbent solution
x_2
x_1
Solution of relaxation is not quite feasible, value = 1.854

Domain filtering with Lagrange multipliers

- So far, this is all standard in global solvers.
- We can achieve stronger propagation with **filtering based on Lagrange multipliers**.
    - Reduced-cost variable fixing is a special case.

Domain filtering with Lagrange multipliers

$$\begin{aligned}
&\min\ x_1 + x_2 \\
&4y = 1 \\
&2x_1 + x_2 \le 2 \\
&\underline{x}_2 x_1 + \underline{x}_1 x_2 - \underline{x}_1 \underline{x}_2 \le y \le \underline{x}_2 x_1 + \bar{x}_1 x_2 - \bar{x}_1 \underline{x}_2 \\
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&\underline{x}_j \le x_j \le \bar{x}_j,\quad j = 1,2
\end{aligned}$$

Associated Lagrange multiplier in solution of relaxation is $\lambda_2 = 1.1$

Domain filtering with Lagrange multipliers

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&\underline{x}_j \le x_j \le \overline{x}_j, \quad j = 1,2
\end{aligned}$$

Associated Lagrange multiplier in solution of relaxation is $\lambda_2 = 1.1$

This yields a valid inequality for propagation:

$$2x_1 + x_2 \ge 2 - \frac{\boxed{1.854} - \boxed{1.25}}{\boxed{1.1}} = 1.451$$

Value of relaxation

Lagrange multiplier

Value of incumbent solution

Domain filtering with Lagrange multipliers

In general, suppose we have a relaxation:

$$\begin{aligned} &\min\ f(x) \\ &g(x) \geq 0 \\ &x \in S \end{aligned}$$

with optimal solution x^*, optimal value v^*, and **Lagrangean dual** solution λ^*.

with $\lambda_i^* > 0$, and U an upper bound on the optimal value of the original problem (perhaps from an incumbent solution).

Then we have the inequality $g_i(x) \leq \dfrac{U - v^*}{\lambda_i^*}$

…which can be propagated.

Domain filtering with Lagrange multipliers

A special case applies to **individual variables**:

$$\begin{aligned} \min\ & f(x) \\ & g(x) \geq 0 \\ & x \in S \end{aligned}$$

has optimal solution x^*, optimal value v^*, and **reduced gradient** r.

with $x_j^* = 0$, and U an upper bound on the optimal value of the original problem (perhaps from an incumbent solution).

Then we have the inequality $$x_j \leq \frac{U - v^*}{r_j}$$

…which fixes $x_j = 0$ if bound < 1 and x_j is integer (**reduced cost variable fixing**)

Piecewise linear functions

- Piecewise linear approximation is a powerful tool for nonlinear optimization.
  - Particularly if nonlinearities are additively separable: $f(x) = \sum_j f_j(x_j)$
- However, MINLP models require auxiliary variables.
  - A serious limitation.

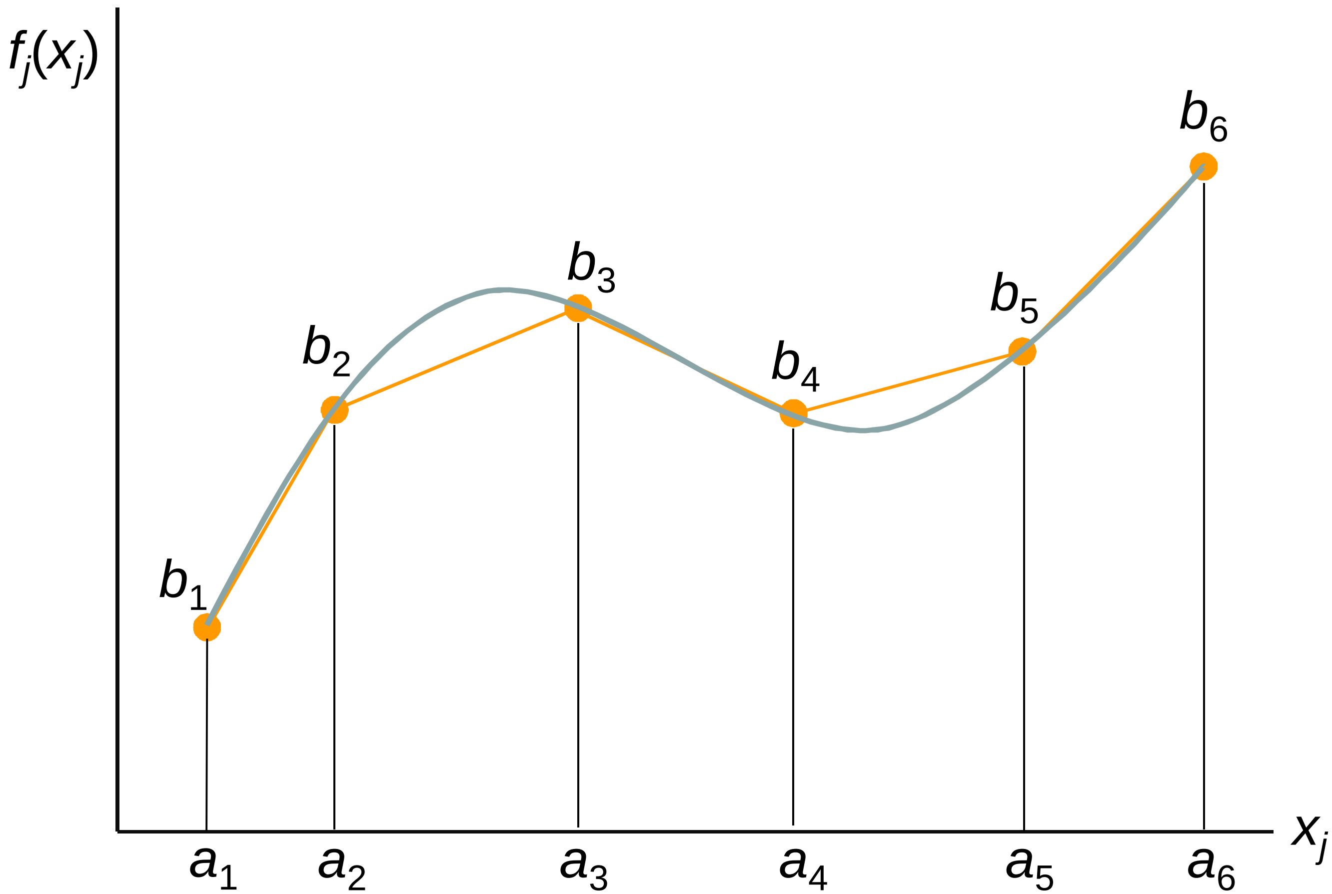
Piecewise linear functions

- CP approach **adds no variables**.
  - while providing convex hull relaxation (tight as any locally ideal MILP model)
- Use piecewise linear **global constraint** for $f_j(x_j)$:

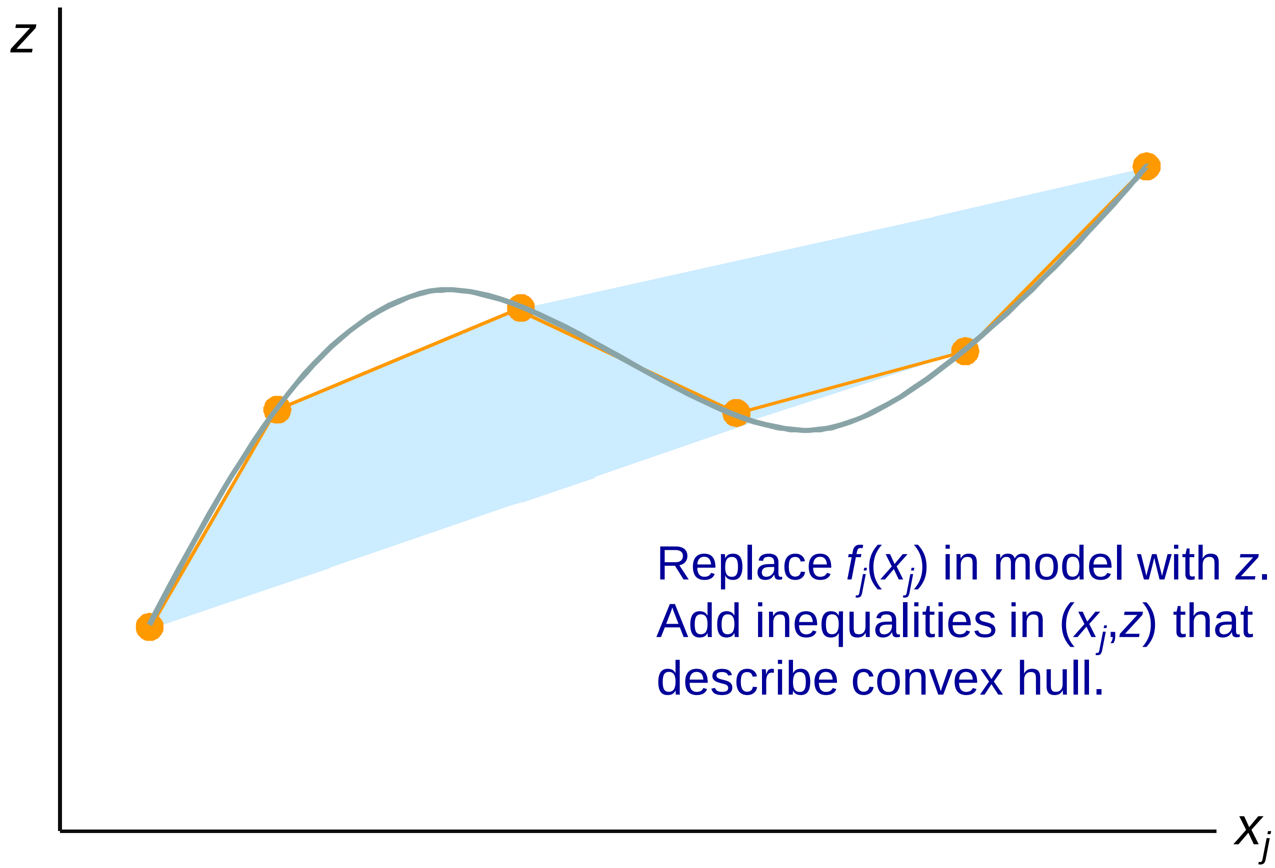
$$\text{piecewise}\left(f_j, x_j, a, b\right)$$

  - where breakpoints are $a = (a_1,\ldots a_n)$ with $f_j(a_i) = b_i$.

Piecewise linear functions



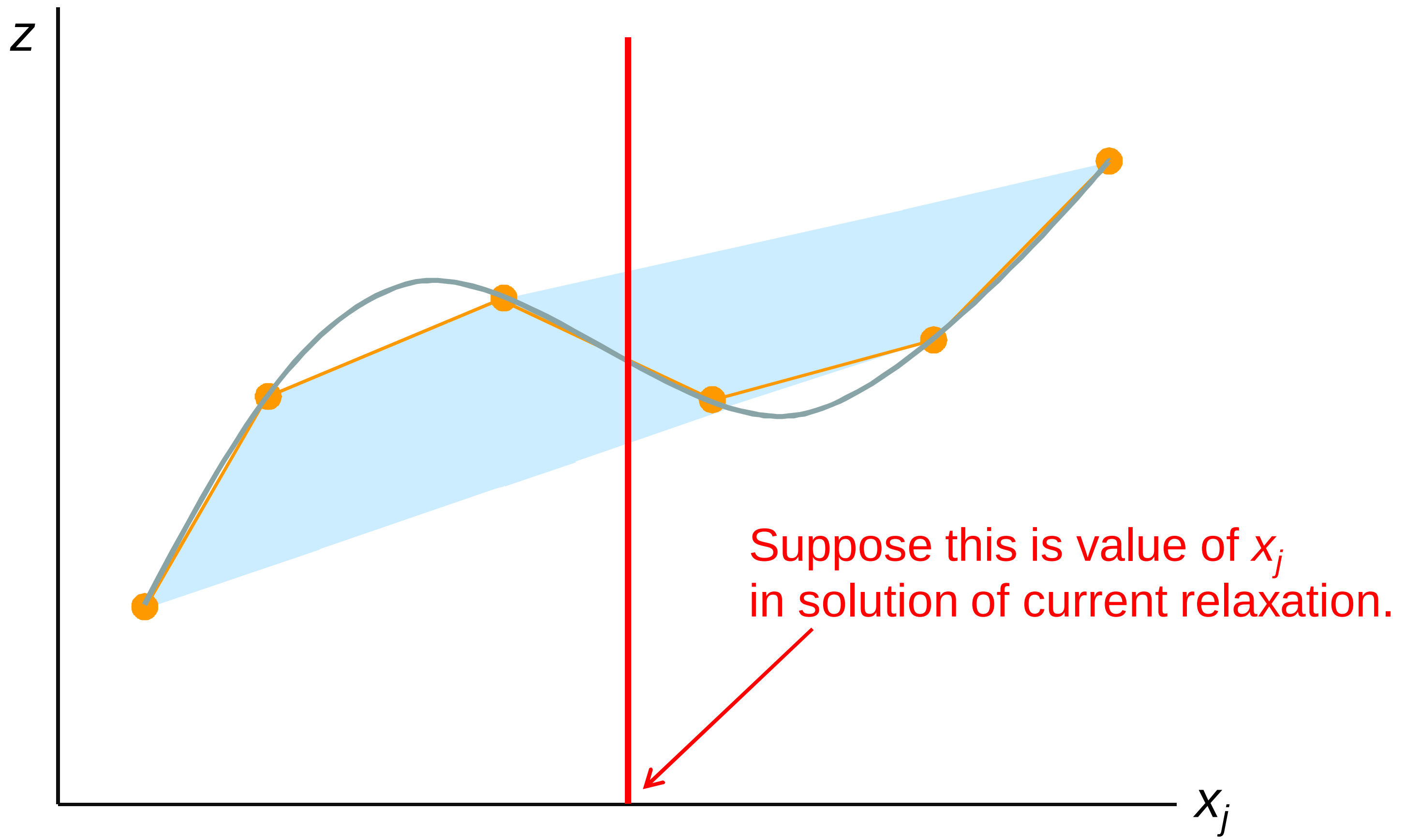
Piecewise linear functions



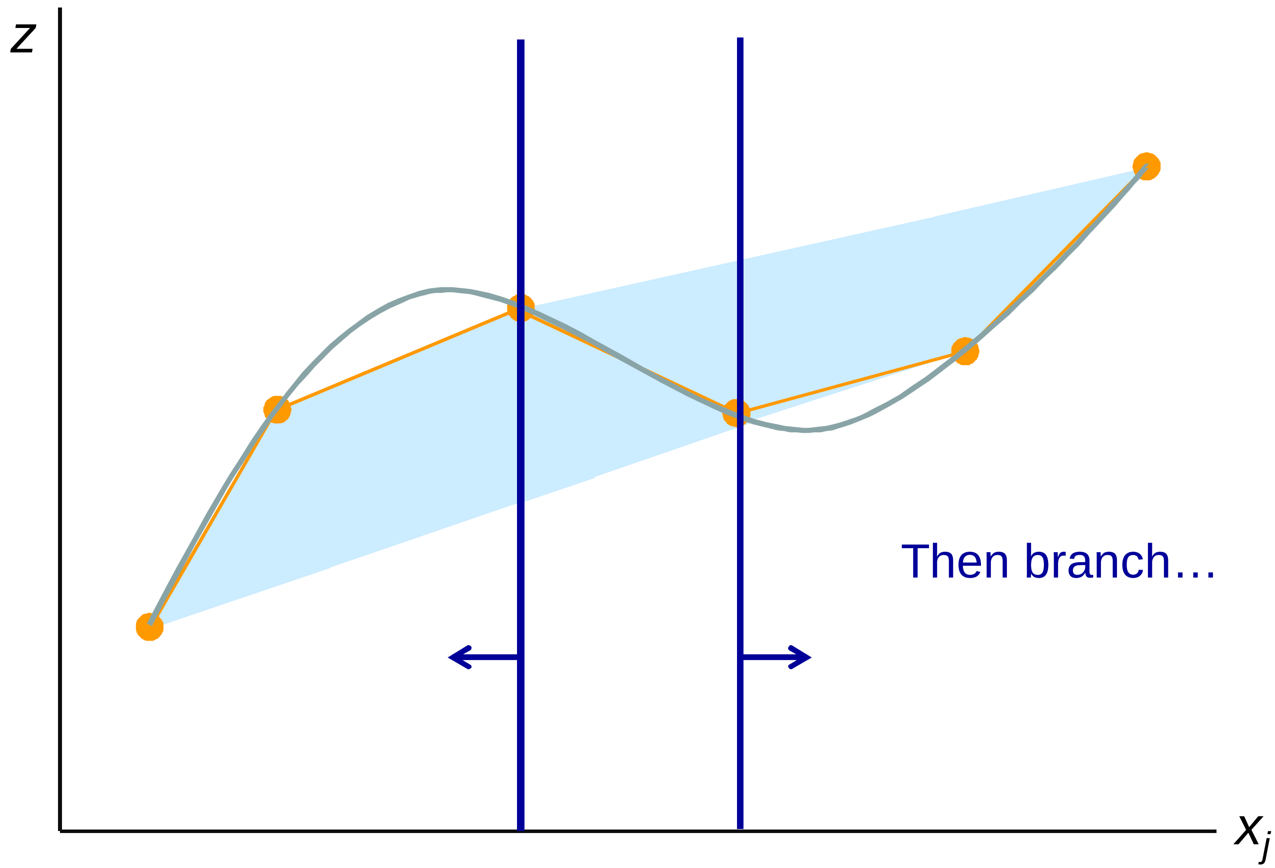
Replace $f_j(x_j)$ in model with z.
Add inequalities in (x_j,z) that describe convex hull.

Convex hull can be computed very rapidly.

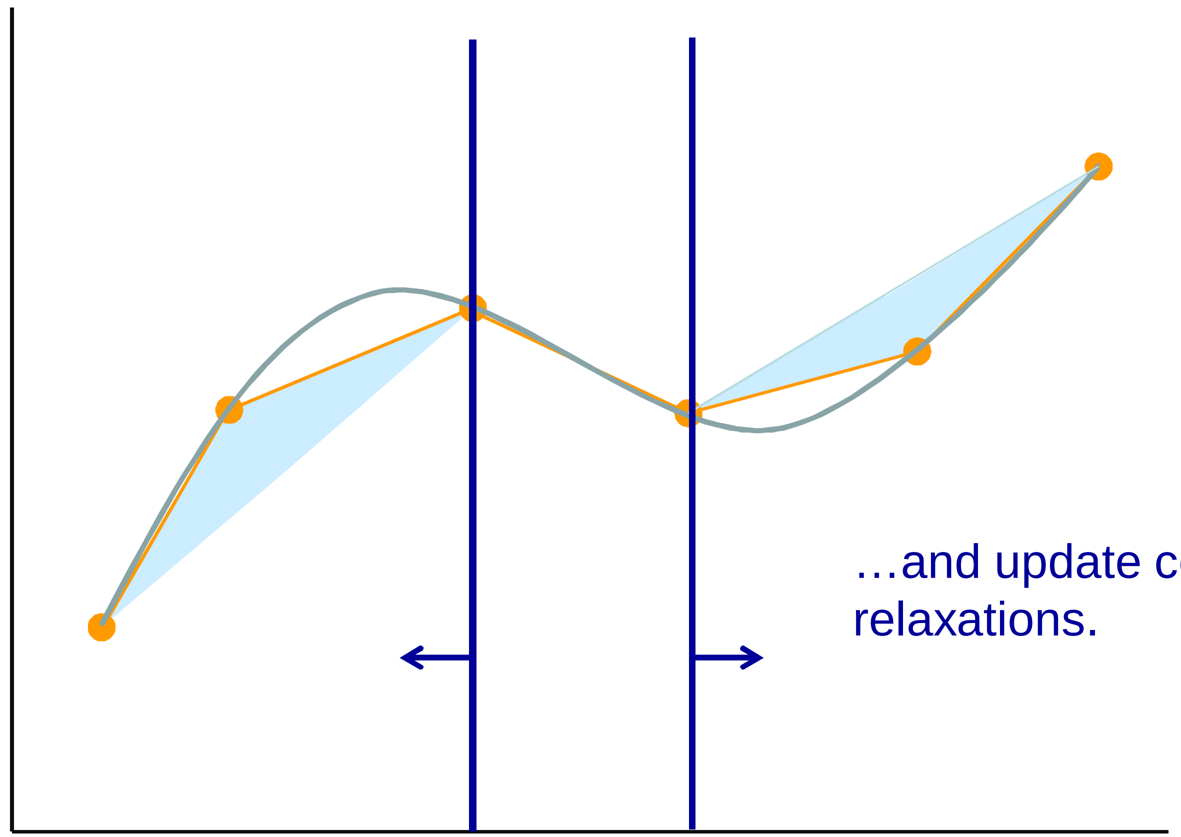
Piecewise linear functions



Piecewise linear functions

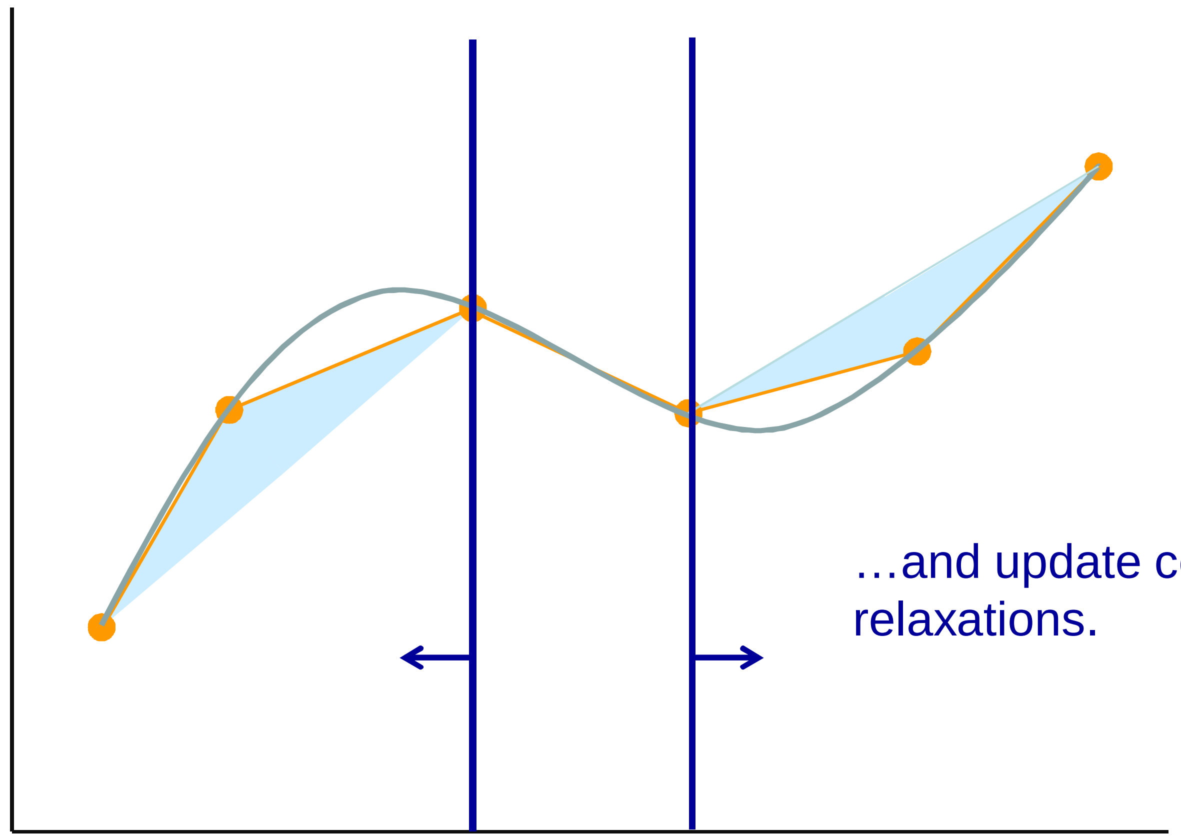


Piecewise linear functions



…and update convex hull relaxations.

Piecewise linear functions



…and update convex hull relaxations.

Easily extended to functions $f_j(x_i,x_k)$
By computing 3D convex hull.

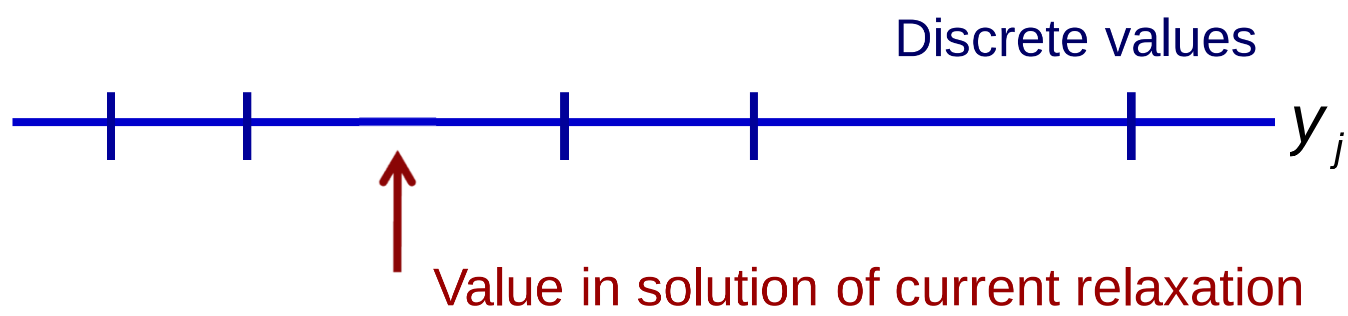
Branching

- In general, branch on discrete values of a variable.
  - …rather than introduce 0-1 variables to model discrete values.
- For a troublesome continuous variable, **discretize it** and branch.
  - Use **many break points** without increasing size of model.

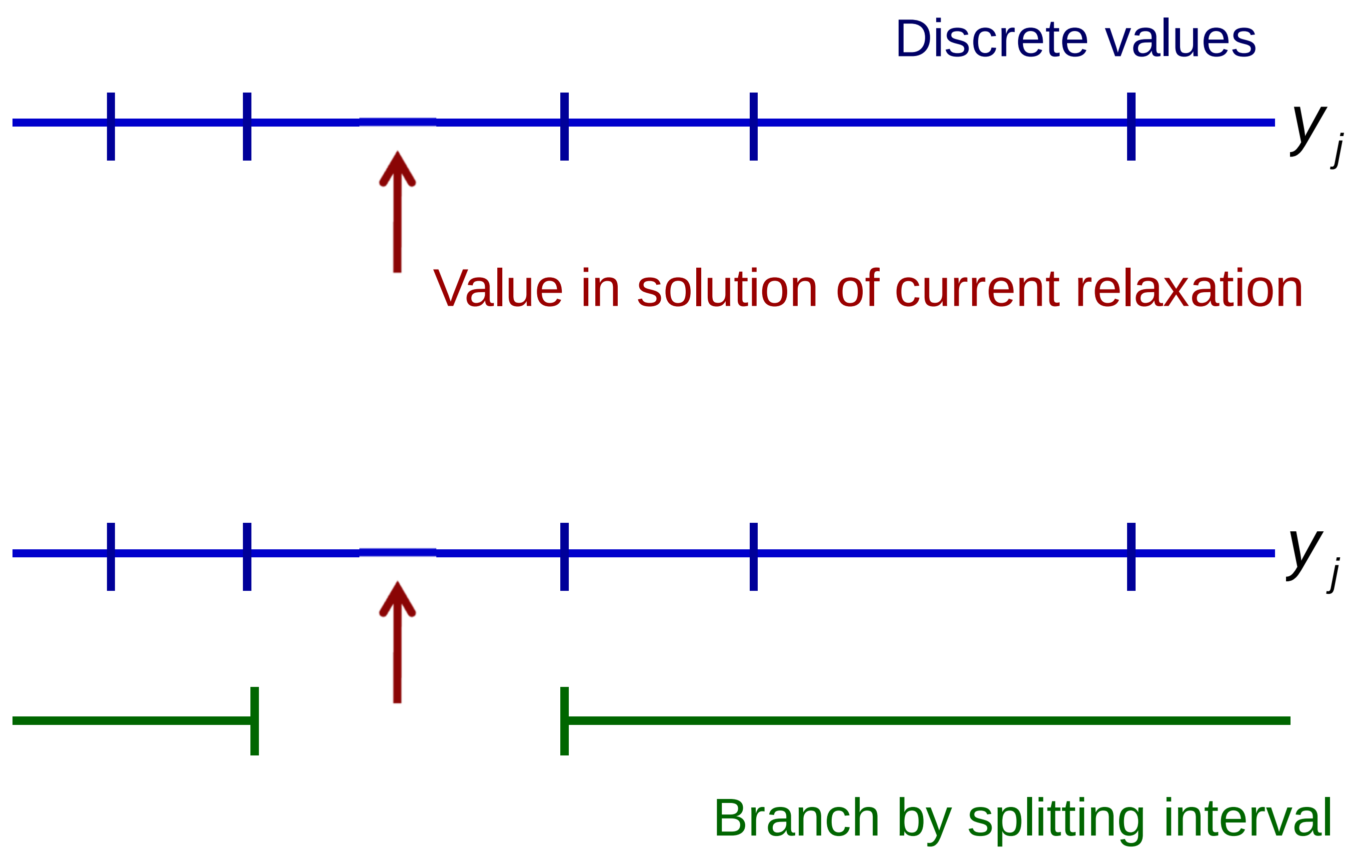
Branching

- In general, branch on discrete values of a variable.
  - …rather than introduce 0-1 variables to model discrete values.
- For a troublesome continuous variable, **discretize it** and branch.
  - Use **many break points** without increasing size of model.
- This may allow for a **convex "**relaxation" (actually, **quasi-relaxation**)
  - If the model becomes convex when discretized variables are fixed.
  - A quasi-relaxation is not a valid relaxation but yields a valid bound on the objective function.

Branching

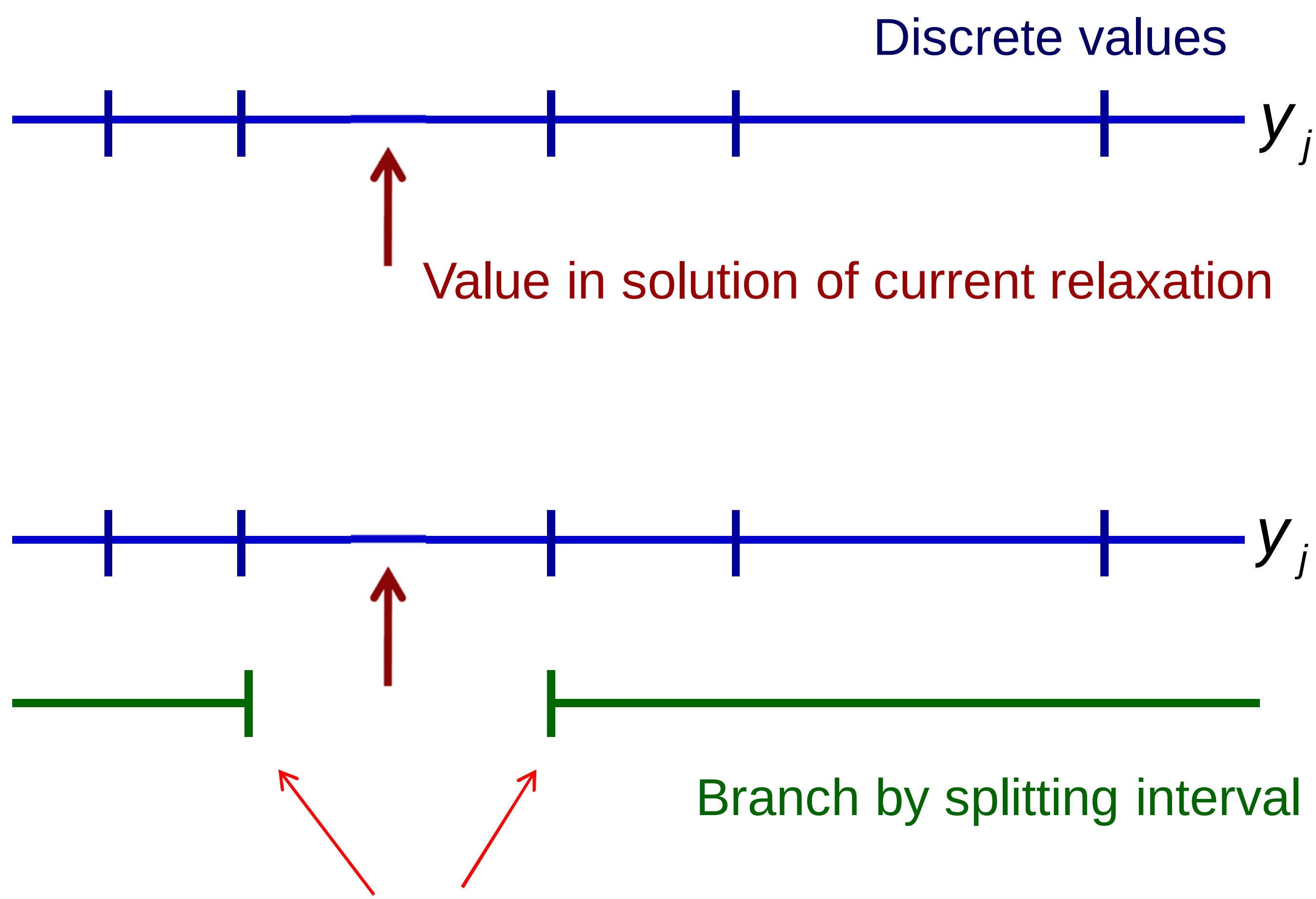

Discrete values
y_j
Value in solution of current relaxation

Branching



Discrete values
y_j
Value in solution of current relaxation
y_j
Branch by splitting interval

Branching



Solution of next relaxation likely to be at an endpoint.
This branching intelligence unavailable in 0-1 model.

Quasi-relaxation

Given problem $\min_{x \in S} \{f(x)\}$

The problem $\min_{x \in S'} \{f'(x)\}$ is a **quasi-relaxation** if for any $x \in S$, there is an $x' \in S'$ with $f'(x') \leq f(x)$.

A quasi-relaxation need not be a valid relaxation.

But its **optimal value** is a **valid lower bound** on the optimal value of the original problem.

Quasi-relaxation

Consider the problem

$$\begin{aligned} &\min\ f(x) \\ &g^j(x, y_j) \le 0, \ \text{all } j \\ &x \in \mathbb{R}^n,\ y_j \ \text{discrete} \end{aligned}$$

Quasi-relaxation

Consider the problem

$$\begin{aligned} &\min\ f(x) \\ &\boxed{g^j}(x, y_j) \le 0, \ \text{all } j \\ &x \in \mathbb{R}^n, y_j \ \text{discrete} \end{aligned}$$

Each g^j is a vector of functions

Quasi-relaxation

Consider the problem

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Each g^j is a vector of functions

Each y_j is a scalar variable

Quasi-relaxation

Consider the problem

$$\begin{aligned} &\min\ f(x) \\ &\boxed{g^j}(x, y_j) \le 0, \text{ all } j \\ &x \in \mathbb{R}^n, \boxed{y_j} \text{ discrete} \end{aligned}$$

Each g^j is a vector of functions

Each y_j is a scalar variable

Relaxing the problem by making y_j continuous could result in a **nonconvex** problem.

Quasi-relaxation

Consider the problem

$$\begin{aligned} &\min\ f(x) \\ &\boxed{g^j}(x, y_j) \le 0,\ \text{all } j \\ &x \in \mathbb{R}^n,\ \boxed{y_j}\ \text{discrete} \end{aligned}$$

Each g^j is a vector of functions

Each y_j is a scalar variable

Relaxing the problem by making y_j continuous could result in a **nonconvex** problem.

But suppose the problem becomes convex when each y_j is fixed to a **constant**.

Quasi-relaxation

Consider the problem

$$\begin{aligned} &\min\ f(x) \\ &\boxed{g^j}(x, y_j) \le 0,\ \text{all } j \\ &x \in \mathbb{R}^n,\ \boxed{y_j}\ \text{discrete} \end{aligned}$$

Each g^j is a vector of functions

Each y_j is a scalar variable

Relaxing the problem by making y_j continuous could result in a **nonconvex** problem.

But suppose the problem becomes convex when each y_j is fixed to a **constant**.

Then we may be able to write a **convex quasi-relaxation**.

Quasi-relaxation

Consider the problem

$$\begin{aligned} &\min\ f(x) \\ &g^j(x, y_j) \le 0,\ \text{all } j \\ &x \in \mathbb{R}^n,\ y_j \text{ discrete} \end{aligned}$$

Theorem (JNH)

If $f(x)$ is convex and each $g^j(x,y)$ is **semihomogeneous** in x and **concave** in scalar y_j, then we have a **convex quasi-relaxation**:

$$\begin{aligned} &\min\ f(x) \\ &g(x^1, y_L) + g(x^2, y_U) \le 0 \\ &\alpha x^L \le x^1 \le \alpha x^U \\ &(1-\alpha)x^L \le x^2 \le (1-\alpha)x^U \\ &x = x^1 + x^2,\ \alpha \in [0,1] \end{aligned}$$

Quasi-relaxation

Consider the problem

$$\begin{aligned}&\min\ f(x)\\ &g^j(x,y_j)\le 0,\ \text{all } j\\ &x\in\mathbb{R}^n,\ y_j\ \text{discrete}\end{aligned}$$

Theorem

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$g(\alpha x,y)\le \alpha g(x,y)$ for all x, y and $\alpha\in[0,1]$,

$g(0,y)=0$ for all y

Quasi-relaxation

Consider the problem

$$\begin{aligned} &\min\ f(x) \\ &g^j(x, y_j) \le 0, \text{ all } j \\ &x \in \mathbb{R}^n,\ y_j \text{ discrete} \end{aligned}$$

Theorem

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Bounds on y

Quasi-relaxation

Consider the problem

$$\begin{aligned} &\min\ f(x) \\ &g^j(x, y_j) \le 0, \text{ all } j \\ &x \in \mathbb{R}^n,\ y_j \text{ discrete} \end{aligned}$$

Theorem

If $f(x)$ is convex and each $g^j(x,y)$ is **semihomogeneous** in x and **concave** in scalar y_j, then we have a **convex quasi-relaxation**:

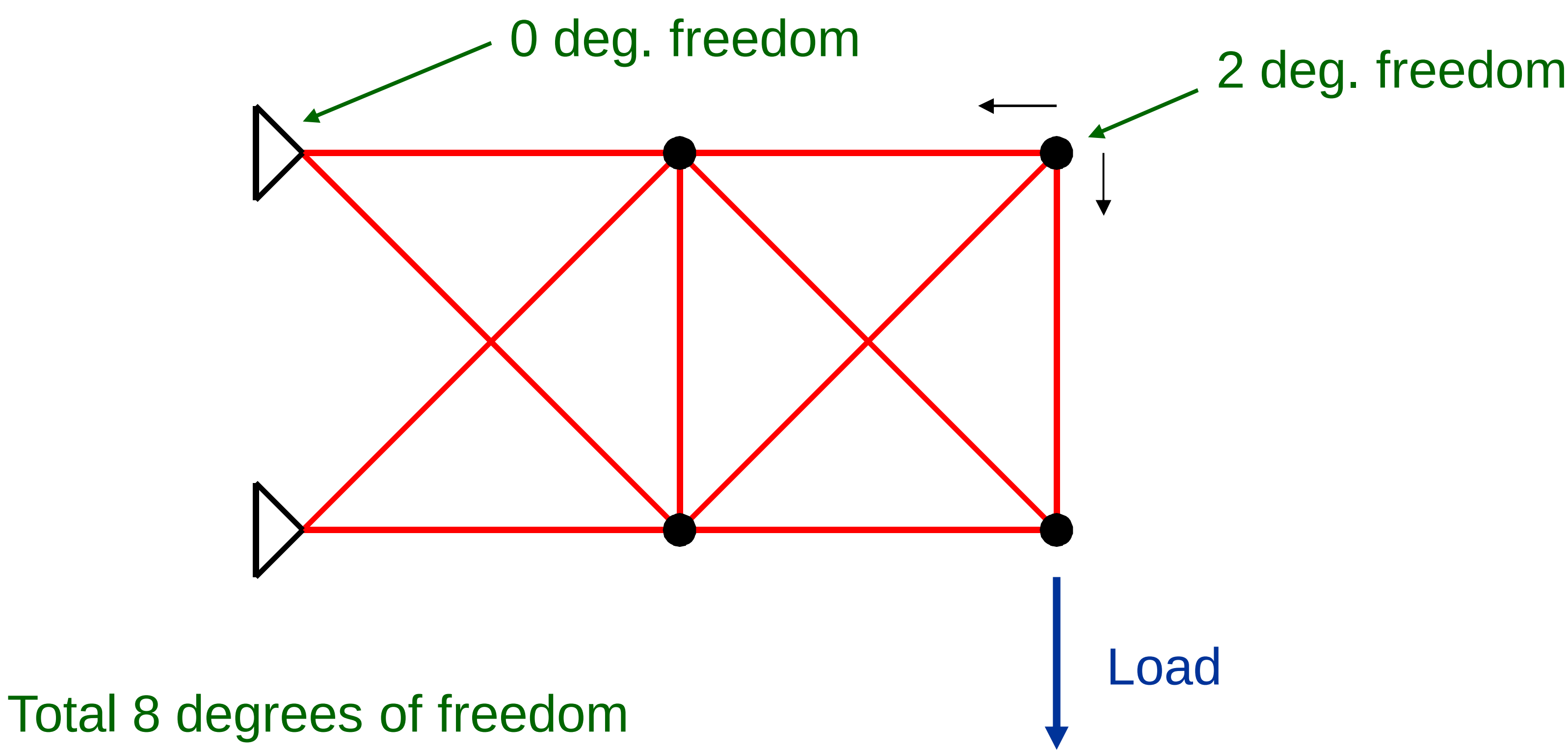
$$\begin{aligned} &\min\ f(x) \\ &g(x^1, y_L) + g(x^2, y_U) \le 0 \\ &\alpha \boxed{x^L} \le x^1 \le \alpha \boxed{x^U} \\ &(1-\alpha)x^L \le x^2 \le (1-\alpha)x^U \\ &x = x^1 + x^2,\ \alpha \in [0,1] \end{aligned}$$

Bounds on x

Example: Truss Structure Design

Select size of each bar (possibly zero) to support the load while minimizing weight. Bar sizes are **discrete**.

10-bar cantilever truss



Truss Structure Design

$$
\begin{aligned}
\min \quad & \sum_i h_i A_i \\
s.t. \quad & \sum_i \cos\theta_{ij} s_i = p_j, \text{ all } j \\
& \sum_j \cos\theta_{ij} d_j = v_i, \text{ all } i \\
& \frac{E_i}{h_i} \boxed{A_i v_i} = s_i, \text{ all } i \\
& v_i^L \le v_i \le v_i^U, \text{ all } i \\
& d_j^L \le d_j \le d_j^U, \text{ all } j \\
& \bigvee_k (A_i = A_{ik})
\end{aligned}
$$

Nonlinear, nonconvex

Hooke's law

Area must be one of several discrete values A_{ik}

Truss Structure Design

Can convert to MILP model by introducing new variables.

$$
\begin{aligned}
\min \quad & \sum_i h_i \sum_k A_{ik} y_{ik} \\
s.t. \quad & \sum_i \cos\theta_{ij} s_i = p_j, \text{ all } j \\
& \sum_j \cos\theta_{ij} d_j = \sum_k v_{ik}, \text{ all } i \\
& \frac{E_i}{h_i} \sum_k A_{ik} v_{ik} = s_i, \text{ all } i \\
& v_i^L \le v_i \le v_i^U, \text{ all } i \\
& d_j^L \le d_j \le d_j^U, \text{ all } j \\
& \sum_k y_{ik} = 1, \text{ all } i
\end{aligned}
$$

0-1 variables indicating size of bar *i*

Elongation variable disaggregated by bar size

Hooke's law becomes linear

Quasi-relaxation

$$
\begin{aligned}
&\min\ f(x) \\
&g^j(x, y_j) \le 0, \ \text{all } j \\
&x \in \mathbb{R}^n,\ y_j \ \text{discrete}
\end{aligned}
$$

$\dfrac{E_i}{h_i} A_i v_i = s_i$ has the form $g(x,y_j) = 0$ with g semihomogeneous in x and concave (linear) in y_j because we can write it

$$\frac{E_i}{h_i} A_i v_i - s_i = 0$$

with $x = (A_i, s_i)$, $y_j = v_i$.

Truss Structure Design

So we have a quasi-relaxation of the truss problem:

$$
\begin{aligned}
\min \quad & \sum_i h_i [A_i^L y_i + A_i^U (1 - y_i)] \\
s.t. \quad & \sum_i \cos\theta_{ij} s_i = p_j, \text{ all } j \\
& \sum_j \cos\theta_{ij} d_j = v_{i0} + v_{i1}, \text{ all } i \\
& \frac{E_i}{h_i}(A_i^L v_{i0} + A_i^U v_{i1}) = s_i, \text{ all } i \\
& v_i^L y_i \le v_{i0} \le v_i^U y_i, \text{ all } i \\
& v_i^L (1 - y_i) \le v_{i1} \le v_i^U (1 - y_i), \text{ all } i \\
& d_j^L \le d_j \le d_j^U, \text{ all } j \\
& 0 \le y_i \le 1, \text{ all } i
\end{aligned}
$$

Hooke's law is linearized

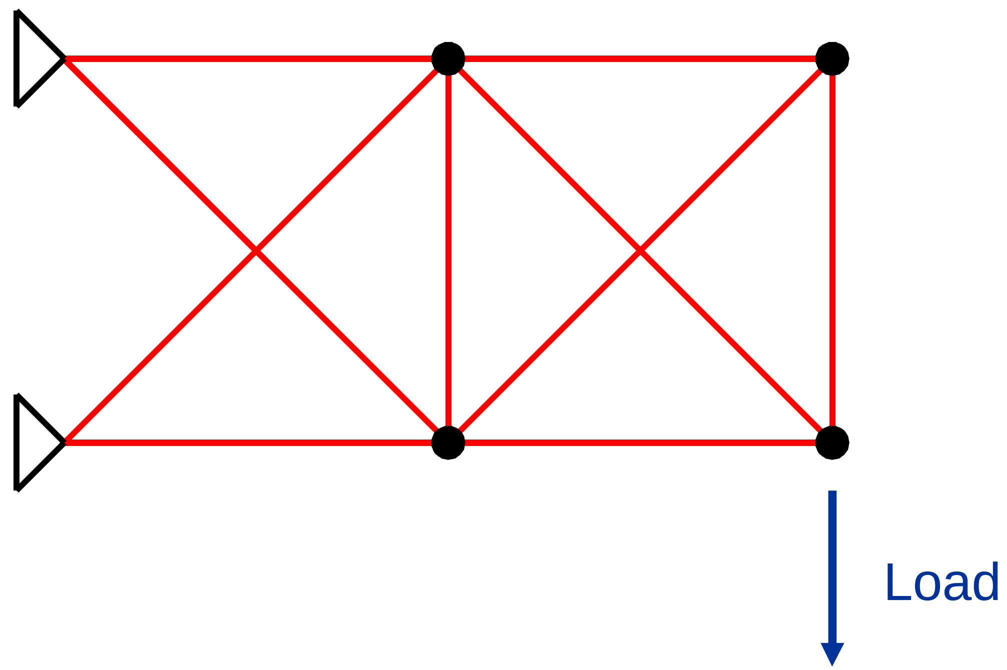
Elongation bounds split into 2 sets of bounds

Truss Structure Design

Some computational results…

10-bar cantilever truss

Yunes, Aron, JNH (2010),
based on
Bollapragada, Ghattas, JNH (2001)



Truss Structure Design

SIMPL = integrated solver that implements CP-style branching and quasi-relaxations

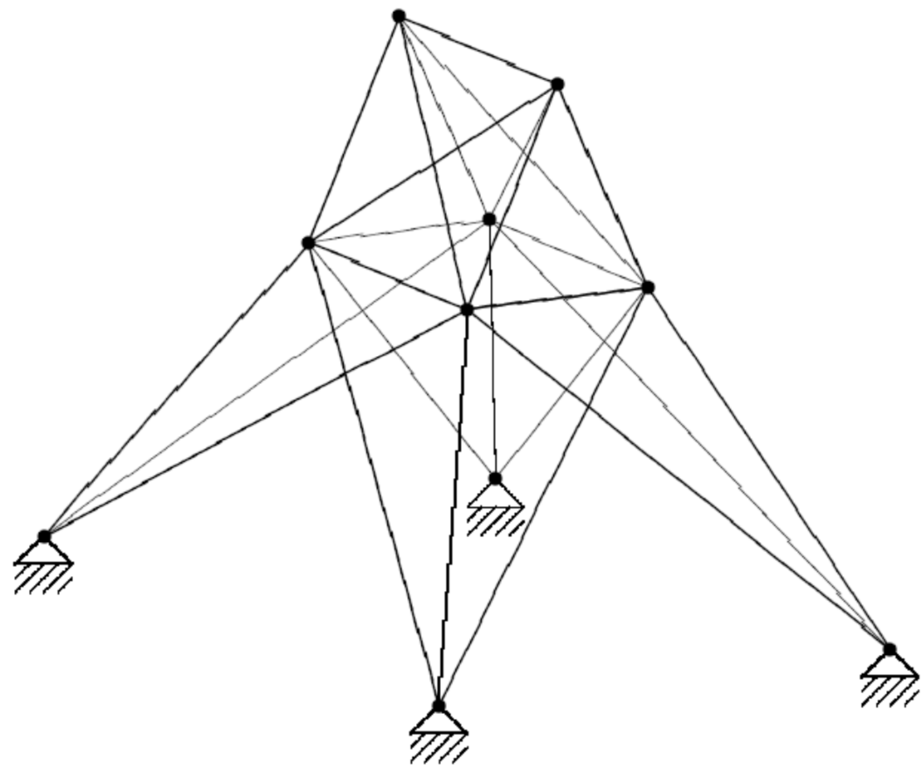
Computational results (seconds)

No. bars	Loads	BARON	CPLEX	SIMPL
10	1	5.3	0.40	0.08
10	1	3.8	0.26	0.07
10	1	8.1	0.83	0.49
10	1	8.8	1.2	0.63
10	2	24	4.9	1.84
10	2*	327	146	65
10	2*	2067	1087	651

*plus displacement bounds

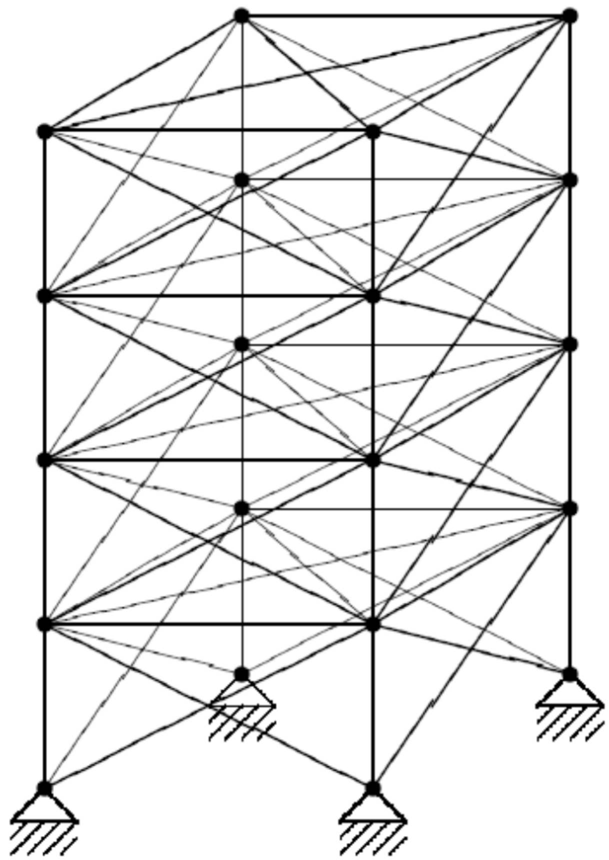
Truss Structure Design

25-bar problem



Truss Structure Design

72-bar problem



Truss Structure Design

Computational results (seconds)

No. bars	Loads	BARON	CPLEX	SIMPL
25	2	3,302	44	20
72	2	3,376	208	28
90	2	21,011	570	92
108	2	> 24 hr*	3208	1720
200	2	> 24 hr*	> 24 hr*	> 24 hr**

* no feasible solution found

** best feasible solution has cost 32,700

Decision diagrams

- A **decision diagram** can represent the feasible set of a discrete optimization problem.
  - An optimal solution is a **shortest path** in the diagram.
  - Linearity, convexity **irrelevant**.
  - Provide **enhanced propagation** in a CP context.
- Proposal: **discretize** continuous variables and optimize over a decision diagram.
  - Branching in **relaxed** decision diagrams may permit **massive discretization**.
  - A “big data” technique.

Decision diagrams

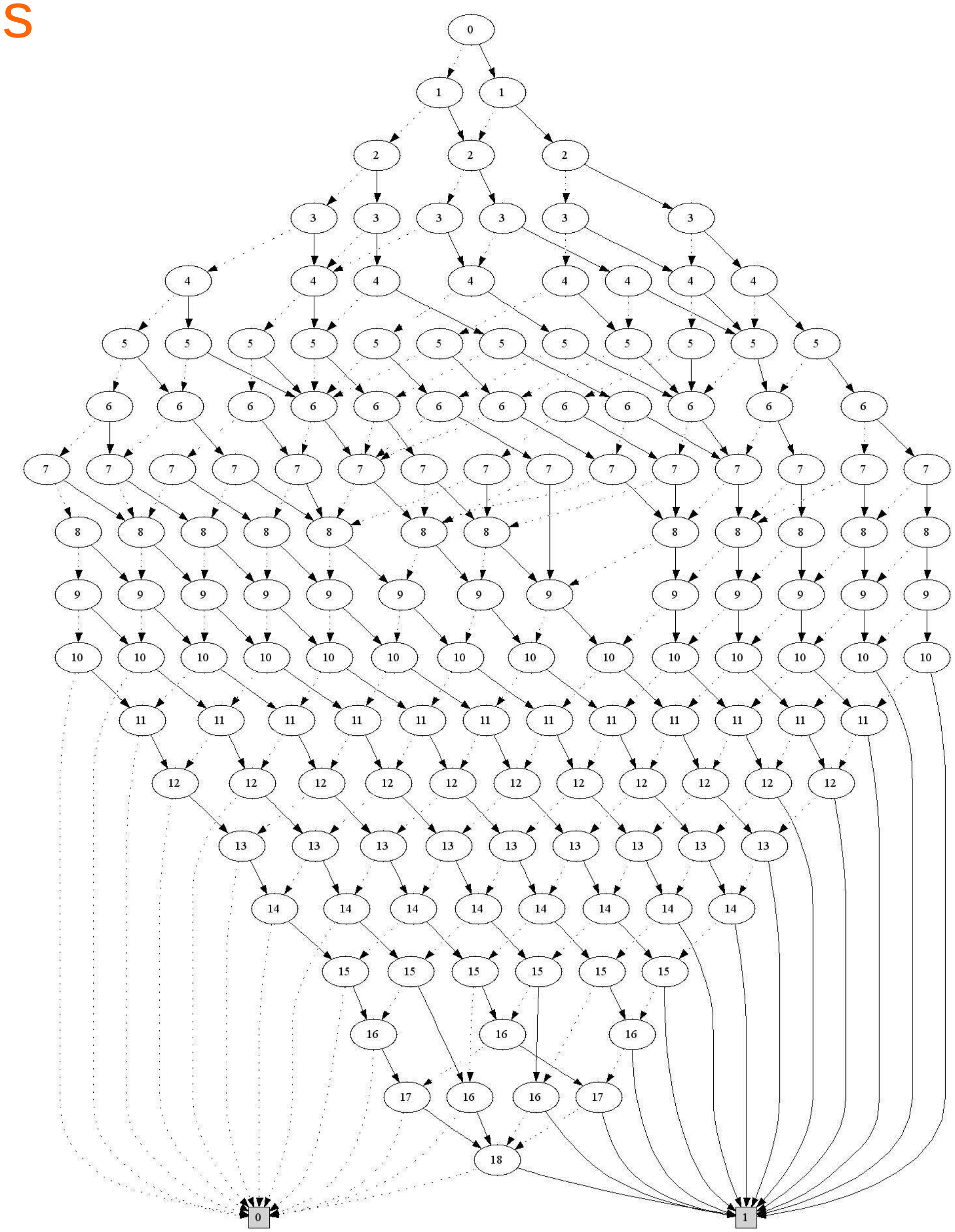
- The knapsack constraint

$$300x_0 + 300x_1 + 285x_2 + 285x_3 + 265x_4 + 265x_5 + 230x_6 + 23x_7 + 190x_8 + 200x_9 + \\ 400x_{10} + 200x_{11} + 400x_{12} + 200x_{13} + 400x_{14} + 200x_{15} + 400x_{16} + 200x_{17} + 400x_{18} \geq 2701$$

has 117,520 minimal feasible solutions.

- But its reduced decision diagram has only 152 nodes…

Decision diagrams

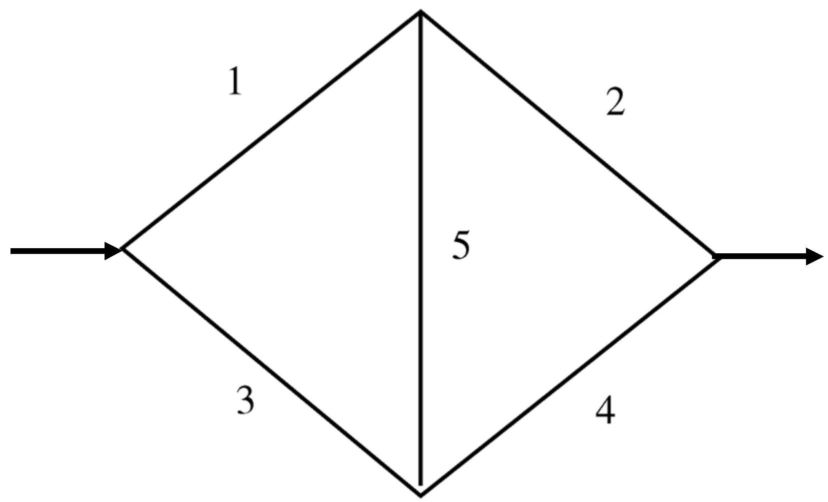


A branch from layer *i* represents fixing x_i to 0 (dashed arc) or 1 (solid arc).

Paths to 1 correspond to feasible solutions.

Example: network reliability

- Minimize cost subject to a bound on reliability (highly nonconvex)
  - System of 5 bridges:



$$R = R_1R_2+(1-R_2)R_3R_4+(1-R_1)R_2R_3R_4+R_1(1-R_2)(1-R_3)R_4R_5+(1-R_1)R_2R_3(1-R_4)R_5$$

Example: network reliability

The problem:

$$\min \sum_j c_j \boxed{x_j}$$

Number of links at bridge *j*

$$R \geq R_{\min}$$

$$R = R_1R_2 + (1-R_2)R_3R_4 + (1-R_1)R_2R_3R_4 + R_1(1-R_2)(1-R_3)R_4R_5 + (1-R_1)R_2R_3(1-R_4)R_5$$

$$R_j = 1-(1-\boxed{r_j})^{x_j}, \ \text{all } j$$

$$x_j \in \{0,1,2,3\}$$

Reliability of one link for bridge *j*

Set min desired reliability to $R_{\min}$ = 60%

Eliminate variables R_i, leaving one **continuous** variable *R*.
Discretize *R* for the decision diagram.

Example: network reliability

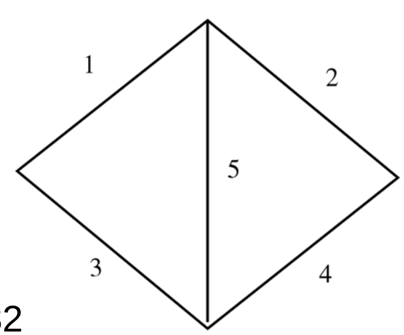
Optimal solution

$c_{opt} + \Delta$	x_1	x_2	x_3	x_4	x_5	R
50:	0	0	1	1	0	72
60:	1	1	0	0,2		79
85:	2					84
90:			2	3		86
95:		2			1	88
100:						95
120:						97
125:	3					
155:		3			2	
160:						98
170:						99
180:			3			
230:					3	

Decision diagram has 308 nodes, generated in 1.1 sec.

Computing optimal solution is trivial (shortest path).

Bonus: we get complete **postoptimality analysis** from decision diagram



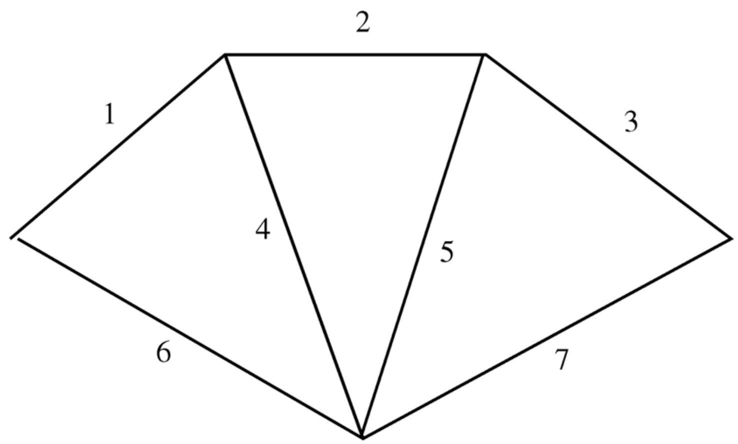
Hadzic and JNH (2006).

Example: network reliability

Nonlinear constraints are increasingly complex for larger networks.

Decision diagram has 1779 nodes, generated in 14.8 sec.

7 bridges

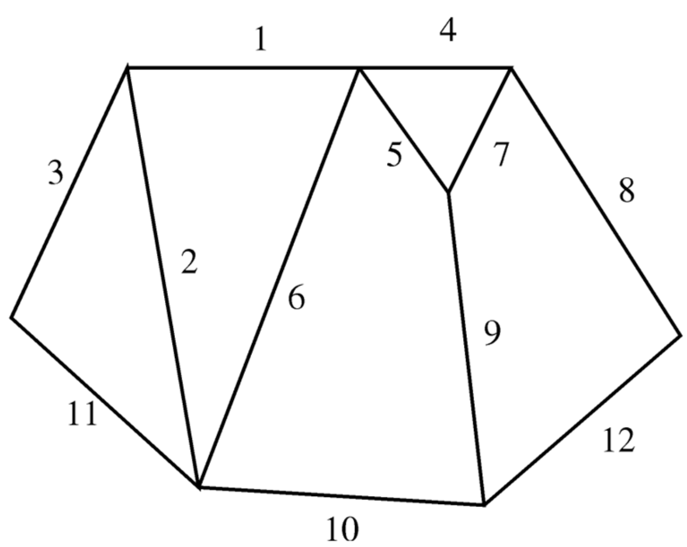


$c_{opt}+\Delta$	x_1	x_2	x_3	x_4	x_5	x_6	x_7	R
9:	0	0	0	0	0	1	1	72.2
11:			1		1		0	
12:	1			1		0		
13:		1				2		82.9
14:					2		2	
15:			2	2				
16:	2							
17:					3	3		84.6
18:		2		3				95.2
19:			3				3	
20:	3							
22:								97.2
23:		3						
27:								99.2
34:								99.4
40:								99.6
43:								99.7
47:								99.8
54:								99.9

Example: network reliability

Decision diagram has 69,457 nodes, generated in 2933 sec.

12 bridges



$c_{opt}+\Delta$	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	x_9	x_{10}	x_{11}	x_{12}	R
180	1	0	2	3	0	0	0	2	0	0	0	0	80
185			3	2									82
190								3					
195													83
200										1			
205													86
210	2						1						
215													88
220										2			
225					1				1				
230	0		0			1,2					1		
235			1										
240		1		1			2			3	2	1	
250				0				0,1				2	
255													91
260	3								2				
265													93
270					2	3	3						
290											3		
300		2											
305									3				
310												3	
315					3								94
340													95
360		3											
365													96
380													97
430													98
485													99

Example: portfolio design

Expected yield rate

Number of blocks of security *i* purchased

$$\max \sum_{i=1}^{n} \mu_i x_i$$

Maximum variance

$$\sum_{i=i}^{n} \sum_{j=1}^{n} c_i c_j \sigma_{ij} x_i x_j \leq V_{max}$$

Maximum investment

$$\sum_{i=1}^{n} c_i x_i \leq W$$

1 if $x_i > 0$,
0 otherwise
(no need for 0-1 variables)

$$\sum_{i=1}^{n} \delta(x_i) \leq K$$

Maximum number of securities in portfolio

$$x_i \in D_i, \; i = 1, \ldots, n$$

Example: portfolio design

10 securities,
max 7 selected.

Decision diagram has 59,802 nodes, generated in 63 sec.

Trivial to compute yield/risk tradeoff.

$c_{opt} - \Delta$	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	x_9	x_{10}
21797	6	0	7	0	3	7	6	0	3	7
21754						6	7			
21705				1					0	6
21683									2	
21678	7		6							
21673					5					
21670					4					
21663				2	0					
21647								2		
21642							5		1	
21630					2					
21624								1,3		
21604					1					
21599						5				
21572			5							
21567									4	
21562										5
21532	5									
21529							4			
21484									5	4
21467						4				
21456								4		
21412					6					
21404							3			
21370		1								
21351								5		
21330									6	
21312			4							
21232						3				
21215										3
21134					7					
21133		2								

Hadzic and JNH (2006).

Decision diagrams

- What if there are **many continuous variables**?
  - Discretize them!
  - Use limited-width **relaxed** decision diagram to obtain optimization bounds.
  - Branch in relaxed decision diagram.
- So far, this method has been applied to IP:
  - Competitive with state-of-the-art IP solvers, or better.
  - Construction of relaxed decision diagram dynamically creates finer granularity for more promising discrete values.

Bergman, Cire, van Hoeve, JNH (2013)

McCormick factorization

- Can be managed with **global constraints** + **semantic typing**.

Cire, JNH, Yunes (2013).

Want to know more about CP and optimization?

- See this websites for links to tutorials (slides & videos):

  http://web.tepper.cmu.edu/jnh/slides.html

- See also:

  http://moya.bus.miami.edu/~tallys/integrated.php (CP + optimization)

  http://www.andrew.cmu.edu/user/vanhoeve/mdd/ (decision diagrams)