Structure in Mixed Integer Conic Optimization: From Minimal Inequalities to Conic Disjunctive Cuts

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Outline

- Mixed integer conic optimization (MICP)
  - Problem setting
  - Motivation
- Structure of linear valid inequalities
  - $\mathcal{K}$-minimal valid inequalities
  - $\mathcal{K}$-sublinear valid inequalities
- Disjunctive cuts for Lorentz cone (joint work with Sercan Yıldız (CMU))
  - Structure of valid linear inequalities
  - A class of valid convex inequalities
  - Nice case and nasty case (with examples)

Problem:

Study the closed convex hull of

$$\mathcal{S}(A,\mathcal{K},\mathcal{B}) = \{x \in E : \; Ax \in \mathcal{B}, \; x \in \mathcal{K}\}$$

where

- E is a finite dimensional Euclidean space with inner product $\langle \cdot, \cdot \rangle$
- A is a linear map from E to $\mathbb{R}^m$
- $\mathcal{B} \subset \mathbb{R}^m$ is a given set of points (can be finite or infinite)
- $\mathcal{K} \subset E$ is a full-dimensional, closed, convex and pointed cone [regular cone]

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Most (if not all!) cutting planes (convexification techniques) in MILPs rely on disjunctive formulations.

Representation Flexibility

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$$\{(y,v) \in \mathbb{R}^k_+ \times \mathbb{Z}^q : \ W\ y + H\ v - b \in \mathcal{K}\}$$

where $\mathcal{K} \subset E$ is a full-dimensional, closed, convex, pointed cone.

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where $\mathcal{K} \subset E$ is a full-dimensional, closed, convex, pointed cone.
Define

$$x = \begin{pmatrix} y \\ z \end{pmatrix}, \quad A = \begin{bmatrix} W, & -\mathrm{Id} \end{bmatrix}, \quad \text{and} \quad \mathcal{B} = b - H\,\mathbb{Z}^q,$$

where Id is the identity map in E. Then we arrive at

$$\mathcal{S}(A,\mathcal{K}',\mathcal{B}) = \{x \in (\mathbb{R}^k \times E) : \ Ax \in \mathcal{B}, \ x \in \underbrace{(\mathbb{R}^k_+ \times \mathcal{K})}_{:=\mathcal{K}'}\}$$

Representation Flexibility

$$\mathcal{S}(A, \mathcal{K}, \mathcal{B}) = \{x \in E : \ Ax \in \mathcal{B}, \ x \in \mathcal{K}\}$$

- Models traditional disjunctive formulations
- Captures the essential non-convex structure of MICPs
- Precisely contains famous Gomory's Corner Polyhedron (1969)
- Arises in modeling complementarity relations
- Can be used as a natural relaxation for sequential convexification
- etc., etc...

$$\mathcal{S}(A,\mathcal{K},\mathcal{B}) = \{x \in E : \; Ax \in \mathcal{B}, \; x \in \mathcal{K}\}$$

General Questions:

- When can we study/characterize $\overline{\text{conv}}(\mathcal{S}(A,\mathcal{K},\mathcal{B}))$ explicitly?
- Will (or when will) $\overline{\text{conv}}(\mathcal{S}(A,\mathcal{K},\mathcal{B}))$ preserve the nice structural properties we originally had?

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- Let's first see what we can find out about the structure of linear valid inequalities...

$$\mathcal{S}(A,\mathcal{K},\mathcal{B}) = \{x \in E : Ax \in \mathcal{B},\ x \in \mathcal{K}\}$$

- $\overline{\text{conv}}(\mathcal{S}(A,\mathcal{K},\mathcal{B}))$ = intersection of all linear valid inequalities (v.i.) $\langle \mu, x\rangle \geq \eta_0$ for $\mathcal{S}(A,\mathcal{K},\mathcal{B})$

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$$\begin{aligned} C(A,\mathcal{K},\mathcal{B}) &= \text{convex cone of all linear valid inequalities for } \mathcal{S}(A,\mathcal{K},\mathcal{B}) \\ &= \left\{ (\mu;\eta_0) : \mu \in E,\ \mu \neq 0,\ -\infty < \eta_0 \leq \inf_{x\in\mathcal{S}(A,\mathcal{K},\mathcal{B})} \langle \mu, x\rangle \right\} \end{aligned}$$

Goal: Study $C(A,\mathcal{K},\mathcal{B})$ in order to characterize the properties of the linear v.i., and identify the necessary and/or sufficient ones defining $\overline{\text{conv}}(\mathcal{S}(A,\mathcal{K},\mathcal{B}))$

Which Inequalities Should We Really Care About?

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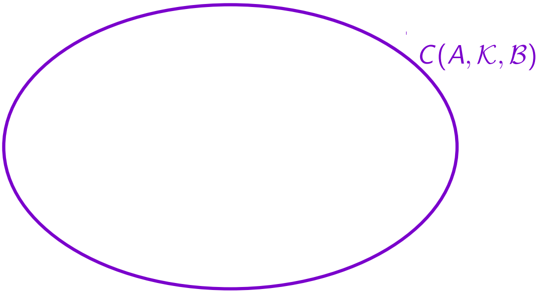
$C(A,\mathcal{K},\mathcal{B}) =$ convex cone of $(\mu;\eta_0)$ of all linear valid inequalities $\langle \mu, x\rangle \geq \eta_0$



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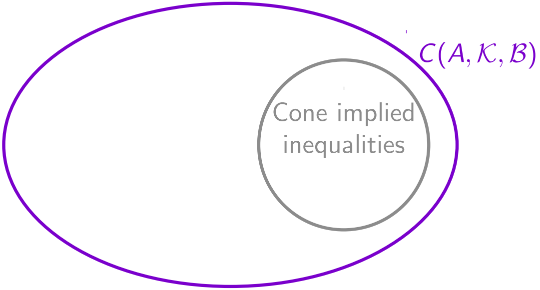
$\Rightarrow$ Cone implied inequality, $(\delta;0)$ for any $\delta \in \mathcal{K}^* \setminus \{0\}$, is always valid.



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Definition

An inequality $(\mu;\eta_0) \in C(A,\mathcal{K},\mathcal{B})$ is a $\mathcal{K}$-minimal valid inequality if for all ρ such that $\rho \preceq_{\mathcal{K}^*} \mu$ and $\rho \neq \mu$, we have $\rho_0 < \eta_0$.

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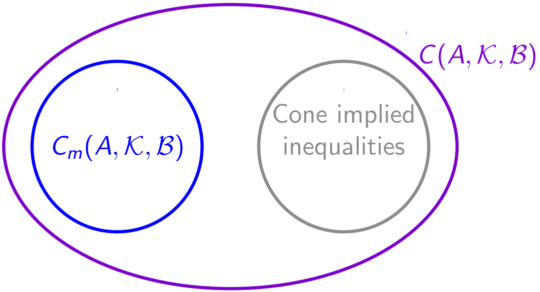

$\mathcal{K}$-minimal Inequalities

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- $\Rightarrow$ Cone implied inequality $(\delta; 0)$ for any $\delta \in \mathcal{K}^* \setminus \{0\}$ is never minimal.
- $\Rightarrow$ $\mathcal{K}$-minimal v.i. exists if and only if $\nexists$ valid equations of form $\langle \delta, x \rangle = 0$ with $\delta \in \mathcal{K}^* \setminus \{0\}$.

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Theorem: [Sufficiency of $\mathcal{K}$-minimal Inequalities]

Whenever $C_m(A, \mathcal{K}, \mathcal{B}) \neq \emptyset$, $\mathcal{K}$-minimal v.i. together with $x \in \mathcal{K}$ constraint are sufficient to describe $\overline{\text{conv}}(\mathcal{S}(A, \mathcal{K}, \mathcal{B}))$.

Some Properties

We can show that

- Every valid inequality $(\mu; \eta_0) \in C(A, \mathcal{K}, \mathcal{B})$ satisfies condition

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Remark

For any $\mu \in \operatorname{Im}(A^*) + \mathcal{K}^*$,

$\Rightarrow$ $D_\mu := \{\lambda \in \mathbb{R}^m : \ \mu - A^*\lambda \in \mathcal{K}^*\} \neq \emptyset$; and

$\Rightarrow$ for any $\eta_0 \leq \inf_{b \in \mathcal{B}} \sigma_{D_\mu}(b)$, where σ_{D_μ} := support function of D_μ,

we have $(\mu; \eta_0) \in C(A, \mathcal{K}, \mathcal{B})$.

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- $\mathcal{K}$-minimal inequalities have more structure, i.e., they are $\mathcal{K}$-sublinear:

Definition

An inequality $(\mu; \eta_0)$ is a $\mathcal{K}$-sublinear v.i. $(C_a(A, \mathcal{K}, \mathcal{B}))$ if it satisfies

$$\textbf{(A.1)} \qquad 0 \le \langle \mu, u \rangle \text{ for all } u \text{ s.t. } Au = 0 \text{ and}$$
$$\langle \alpha, v \rangle u + v \in \mathcal{K} \;\; \forall v \in \mathrm{Ext}(\mathcal{K}) \text{ holds for some } \alpha \in \mathrm{Ext}(\mathcal{K}^*),$$
$$\textbf{(A.2)} \qquad \eta_0 \le \langle \mu, x \rangle \text{ for all } x \in \mathcal{S}(A, \mathcal{K}, \mathcal{B}).$$

Condition **(A.1)** implies **(A.0)**.

- We can establish necessary, and also sufficient conditions for an inequality $(\mu; \eta_0)$ to be $\mathcal{K}$-minimal or $\mathcal{K}$-sublinear via its relation with support function σ_{D_μ} of the structured set D_μ.

On Conditions for $\mathcal{K}$-minimality and $\mathcal{K}$-sublinearity

- We can establish necessary, and also sufficient conditions for an inequality $(\mu; \eta_0)$ to be $\mathcal{K}$-minimal or $\mathcal{K}$-sublinear via its relation with support function σ_{D_μ} of the structured set D_μ.
- When $\mathcal{K} = \mathbb{R}^n_+$:
  - $\mathcal{K}$-sublinear v.i. are identical to the class of subadditive v.i. defined by Johnson'81, i.e., condition (**A.1**) is precisely condition (**A.0**) and

$$\textbf{(A.1i)} \quad \text{for all } i = 1, \ldots, n, \;\; \mu_i \leq \langle \mu, u \rangle \text{ for all } u \in \mathbb{R}^n_+ \text{ s.t. } \;\; Au = A^i.$$

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  - Our sufficient condition for $\mathcal{K}$-sublinearity matches precisely our necessary condition, resulting in

$$(\mu; \eta_0) \in C_a(A, \mathcal{K}, \mathcal{B}) \quad \iff \quad \mu_i = \sigma_{D_\mu}(A^i) \text{ for all } i.$$

$\Rightarrow$ All $\mathbb{R}^n_+$-sublinear (and thus $\mathbb{R}^n_+$-minimal) inequalities are generated by sublinear functions (subadditive and positively homogeneous, in fact also piecewise linear and convex), i.e., support functions $\sigma_{D_\mu}(\cdot)$ of D_μ.

[This recovers a number of results from Johnson'81, and Conforti et al.'13.]

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$\Rightarrow$ This is underlies a cut generating function view for MILPs.

On Conditions for $\mathcal{K}$-minimality and $\mathcal{K}$-sublinearity

- For general regular cones $\mathcal{K}$ other than $\mathbb{R}^n_+$, unfortunately there is a gap between our current necessary condition and our sufficient condition for $\mathcal{K}$-minimality.
- Moreover, there is a simple example $\mathcal{S}(A, \mathcal{K}, \mathcal{B})$ with $\mathcal{K} = \mathcal{L}^3$, where a necessary (in terms of convex hull description) family of $\mathcal{K}$-minimal inequalities cannot be generated by any class of cut generating functions.

  This is in sharp contrast

  - to MILP case, i.e., $\mathcal{K} = \mathbb{R}^n_+$, and,
  - to the strong dual for MICP result of Moran et al.'12, which studied a more restrictive conic setup.

$$\mathcal{S}(A,\mathcal{K},\mathcal{B}) = \{x \in E : \ Ax \in \mathcal{B}, \ x \in \mathcal{K}\}$$

General Questions:

- When can we characterize $\overline{\text{conv}}(\mathcal{S}(A,\mathcal{K},\mathcal{B}))$ explicitly?
- Will (or when will) $\overline{\text{conv}}(\mathcal{S}(A,\mathcal{K},\mathcal{B}))$ preserve the nice structural properties we originally had?

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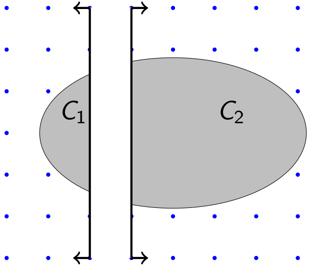
- General case is too general for us to answer these questions...
- In the rest of this talk, we will study a simple (?) yet interesting case
- Joint work with S. Yıldız

- Start with a simple set for x, i.e., a regular cone $\mathcal{K} \subseteq \mathbb{R}^n$
- Consider a two-term disjunction: either $c_1^T x \geq c_{1,0}$ or $c_2^T x \geq c_{2,0}$ must hold.
- Let $C_i := \{x : \ c_i^T x \geq c_{i,0}, \ x \in \mathcal{K}\}$.



A special case is split disjunctions, i.e., $c_1 = -\tau c_2$ for some $\tau > 0$, and $c_{1,0} c_{2,0} > 0$.

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- By setting

$$A = \begin{bmatrix} c_1^T \\ c_2^T \end{bmatrix}, \text{ and } \mathcal{B} = \left\{ \begin{bmatrix} \{c_{1,0}\} + \mathbb{R}_+ \\ \mathbb{R} \end{bmatrix} \bigcup \begin{bmatrix} \mathbb{R} \\ \{c_{2,0}\} + \mathbb{R}_+ \end{bmatrix} \right\}$$

we arrive at

$$\mathcal{S}(A, \mathcal{K}, \mathcal{B}) = \{x \in \mathbb{R}^n : Ax \in \mathcal{B}, \ x \in \mathcal{K}\} = C_1 \cup C_2.$$

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We are interested in describing $\overline{\text{conv}}(\mathcal{S}(A, \mathcal{K}, \mathcal{B}))$ in the original space of variables:

- Is there any structure in $\overline{\text{conv}}(\mathcal{S}(A, \mathcal{K}, \mathcal{B}))$?
- Can we preserve the simple conic structure we started out with?

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$\Rightarrow$ Simple set we start out can be $\mathcal{U} = \{x \in \mathbb{R}^n : \; Qx - d \in \mathcal{K}\}$ with $Q \in \mathbb{R}^{m \times n}$ having full row rank.

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Approach

- Characterize the structure of $\mathcal{K}$-minimal and tight valid linear inequalities
- Using conic duality, we group these linear inequalities appropriately, and thus derive a family of convex valid inequalities sufficient to describe the closed convex hull
- Any structure beyond convexity?
  - $\Rightarrow$ Understand when these convex inequalities are conic representable
- When does a single inequality from this family suffice?
  - $\Rightarrow$ Characterize when only single inequality from this family is sufficient to describe the closed convex hull

Setup for Conic Disjunctive Cuts

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- WLOG we assume that
  - $c_{1,0}, c_{2,0} \in \{0, \pm 1\}$ and
  - $C_1 \neq \emptyset$ and $C_2 \neq \emptyset$ (in fact we assume C_1, C_2 are strictly feasible)
  - $C_1 \not\subseteq C_2$ and $C_2 \not\subseteq C_1$.

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  - $C_1 \neq \emptyset$ and $C_2 \neq \emptyset$ (in fact we assume C_1, C_2 are strictly feasible)
  - $C_1 \not\subseteq C_2$ and $C_2 \not\subseteq C_1$. This assumption is equivalent to

Assumption

The disjunction $c_1^T x \geq c_{1,0}$ or $c_2^T x \geq c_{2,0}$ satisfies

- $\{\beta \in \mathbb{R}_+ : \ \beta c_{1,0} \geq c_{2,0}, \ c_2 - \beta c_1 \in \mathcal{K}\} = \emptyset$, and
- $\{\beta \in \mathbb{R}_+ : \ \beta c_{2,0} \geq c_{1,0}, \ c_1 - \beta c_2 \in \mathcal{K}\} = \emptyset$.

Disjunction on $\mathcal{K}$: either $c_1^T x \geq c_{1,0}$ or $c_2^T x \geq c_{2,0}$ with $c_{1,0}, c_{2,0} \in \{0, \pm 1\}$

Standard Approach

For any valid linear inequality, $\mu^\top x \geq \mu_0$ for $\overline{\text{conv}}(C_1 \cup C_2)$
there exists $\alpha_1, \alpha_2 \in \mathcal{K}$, and $\beta_1, \beta_2 \in \mathbb{R}_+$ s.t.

$$\begin{aligned} \mu &= \alpha_1 + \beta_1 c_1, \\ \mu &= \alpha_2 + \beta_2 c_2, \\ \mu_0 &\leq \min\{\beta_1 c_{1,0}, \beta_2 c_{2,0}\}. \end{aligned}$$

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Proposition

For any $\mathcal{K}$-minimal and tight valid linear inequality, $\mu^\top x \geq \mu_0$ for $\overline{\text{conv}}(C_1 \cup C_2)$ there exists $\alpha_1, \alpha_2 \in \text{bd}(\mathcal{K})$, and $\beta_1, \beta_2 \in (\mathbb{R}_+ \setminus \{0\})$ s.t.

$$\begin{aligned} \mu &= \alpha_1 + \beta_1 c_1, \\ \mu &= \alpha_2 + \beta_2 c_2, \\ \min\{c_{1,0}\beta_1, c_{2,0}\beta_2\} &= \mu_0 = \min\{c_{1,0}, c_{2,0}\}, \end{aligned}$$

and at least one of β_1 and β_2 is equal to 1.

Deriving a Nonlinear (but Convex) Valid Inequality when $\mathcal{K} = \mathcal{L}^n$

Assume $c_{1,0} \geq c_{2,0} \quad \Rightarrow \quad \beta_2 = 1$

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$$\mu \in \mathcal{M}(\beta, 1) := \{\mu \in \mathbb{R}^n : \exists \alpha_1, \alpha_2 \in \text{bd}\, \mathcal{L}^n \ \text{ s.t. } \ \mu = \alpha_1 + \beta c_1 = \alpha_2 + c_2\}$$

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where $M := (\beta^2 \|\tilde{c}_1\|_2^2 - \|\tilde{c}_2\|_2^2) - (\beta^2 c_{1,n}^2 - c_{2,n}^2)$.

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$x \in \overline{\text{conv}}(C_1 \cup C_2) \ \Rightarrow \ x \in \mathcal{L}^n$ and $\mu^\top x \geq c_{2,0} \quad \forall \mu \in \mathcal{M}(\beta, 1)$

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$$\begin{aligned} x \in \overline{\operatorname{conv}}(C_1 \cup C_2) \ &\Rightarrow\ x \in \mathcal{L}^n \text{ and } \mu^\top x \geq c_{2,0} \quad \forall \mu \in \mathcal{M}(\beta,1) \\ &\Leftrightarrow x \in \mathcal{L}^n \text{ and } \inf_{\mu}\left\{\mu^\top x :\ \mu \in \mathcal{M}(\beta,1)\right\} \geq c_{2,0}\end{aligned}$$

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Convex Disjunctive Cut for $\mathcal{K} = \mathcal{L}^n$

Disjunction on second-order cone, $\mathcal{L}^n = \{x \in \mathbb{R}^n : \; x_n \geq \|\widetilde{x}\|_2\}$:
either $c_1^T x \geq c_{1,0}$ or $c_2^T x \geq c_{2,0}$ with $c_{1,0} \geq c_{2,0}$

Theorem

For any $\beta > 0$ s.t. $\beta c_{1,0} \geq c_{2,0}$ and $\beta c_1 - c_2 \notin \pm\mathrm{int}(\mathcal{L}^n)$, then the following inequality is valid for $\overline{\mathrm{conv}}(C_1 \cup C_2)$:

$$2c_{2,0} - (\beta c_1 + c_2)^\top x \leq \sqrt{\left((\beta c_1 - c_2)^\top x\right)^2 + N(\beta) * (x_n^2 - \|\widetilde{x}\|_2^2)}$$

where $N(\beta) := \|\beta \widetilde{c}_1 - \widetilde{c}_2\|_2^2 - (\beta c_{1,n} - c_{2,n})^2$.

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where $N(\beta) := \|\beta\tilde{c}_1 - \tilde{c}_2\|_2^2 - (\beta c_{1,n} - c_{2,n})^2$.

From its construction, this inequality

- exactly captures all undominated linear v.i. $\mu^T x \geq \mu_0$ corresponding to $\beta_1 = \beta$ and $\beta_2 = 1$, e.g., $\mu_0 = c_{2,0}$ and $\mu \in \mathcal{M}(\beta, 1)$

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From its construction, this inequality

- exactly captures all undominated linear v.i. $\mu^T x \geq \mu_0$ corresponding to $\beta_1 = \beta$ and $\beta_2 = 1$, e.g., $\mu_0 = c_{2,0}$ and $\mu \in \mathcal{M}(\beta, 1)$
- is valid and convex
- reduces to the linear inequality $\beta c_1^\top x \geq c_{2,0}$ in $\mathcal{L}^n$ when $\beta c_1 - c_2 \in \pm \, \mathrm{bd} \, \mathcal{L}^n$

Structure of Convex Valid Inequalities

$$2c_{2,0} - (\beta c_1 + c_2)^\top x \leq \sqrt{((\beta c_1 - c_2)^\top x)^2 + N(\beta) * (x_n^2 - \|\tilde{x}\|_2^2)}$$

Any further structure than convexity?

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Any further structure than convexity?

Proposition

An equivalent conic quadratic form given by

$$N(\beta)x + 2(c_2^\top x - c_{2,0}) \begin{pmatrix} \beta\widetilde{c}_1 - \widetilde{c}_2 \\ -\beta c_{1,n} + c_{2,n} \end{pmatrix} \in \mathcal{L}^n$$

is valid whenever a symmetry condition, e.g.,

$$-2c_{2,0} + (\beta c_1 + c_2)^\top x \le \sqrt{((\beta c_1 - c_2)^\top x)^2 + N(\beta)\,(x_n^2 - \|\widetilde{x}\|^2)}$$

holds for all $x \in \overline{\text{conv}}(C_1 \cup C_2)$.

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An equivalent conic quadratic form is valid, e.g., symmetry condition holds, when

- $C_1 \cap C_2 = \emptyset$, i.e., a proper split disjunction, or
- $\{x \in \mathcal{L}^n : \ \beta c_1^\top x \ge c_{2,0}, \ c_2^\top x \ge c_{2,0}\} = \{x \in \mathcal{L}^n : \ \beta c_1^\top x = c_{2,0}, \ c_2^\top x = c_{2,0}\}$

When does a Single Convex Inequality Suffice?

A parametric family of convex inequalities:

For any $\beta > 0$ s.t. $\beta c_{1,0} \geq c_{2,0}$ and $\beta c_1 - c_2 \notin \pm \mathrm{int}(\mathcal{L}^n)$,

$$2c_{2,0} - (\beta c_1 + c_2)^\top x \leq \sqrt{((\beta c_1 - c_2)^\top x)^2 + N(\beta) * (x_n^2 - \|\tilde{x}\|_2^2)}$$

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is valid for $\overline{\mathrm{conv}}(C_1 \cup C_2)$.

Theorem

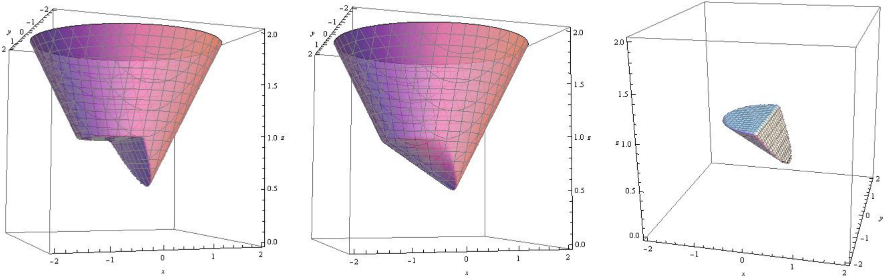
In certain cases such as

- $c_1 \in \mathcal{L}^n$ or $c_2 \in \mathcal{L}^n$, or
- $c_{1,0} = c_{2,0} \in \{\pm 1\}$ and $\mathrm{conv}(C_1 \cup C_2)$ is closed, e.g., in the case of split disjunctions,

it is sufficient (for $\overline{\mathrm{conv}}(C_1 \cup C_2)$) to consider only one inequality with $\beta = 1$.

Example: Nice Case

Disjunction: $\quad x_3 \geq 1 \quad$ or $\quad x_1 + x_3 \geq 1$



$$\overline{\mathrm{conv}}(C_1 \cup C_2) = \left\{ x \in \mathcal{L}^3 : \; 2 - (x_1 + 2x_3) \leq \sqrt{x_3^2 - x_2^2} \right\}$$

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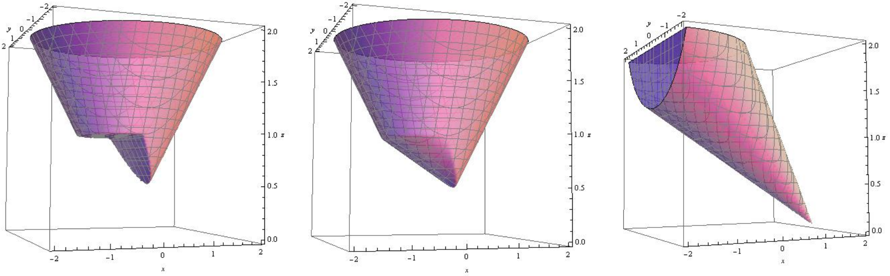

$$\overline{\text{conv}}(C_1 \cup C_2) = \left\{ x \in \mathcal{L}^3 : \, 2 - (x_1 + 2x_3) \leq \sqrt{x_3^2 - x_2^2} \right\}$$

Not So Nice Case

There are cases when we need infinitely many convex inequalities from this family.

- Recessive directions, and
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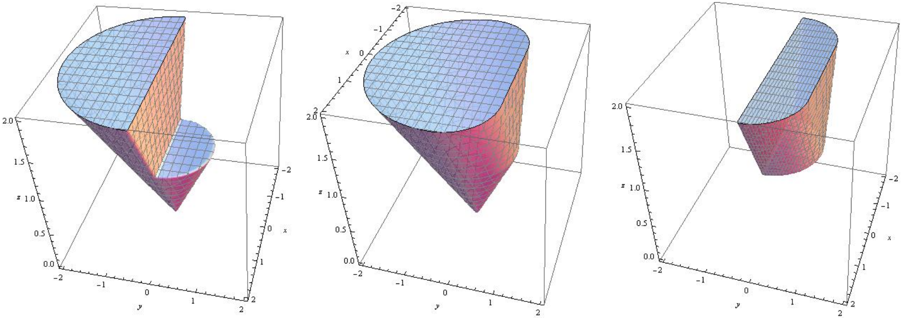
- $\mathrm{conv}(C_1 \cup C_2)$ being non-closed,

play a key role in these cases.

We can still give expressions for a single inequality describing $\overline{\mathrm{conv}}(C_1 \cup C_2)$, but it is really nasty looking...

Example: Nasty Case

Disjunction: $\quad -x_3 \geq -1 \quad \text{or} \quad -x_2 \geq 0$



$$\overline{\text{conv}}(C_1 \cup C_2) = \left\{ x \in \mathcal{L}^3 : \; x_2 \leq 1, \; 1 + |x_1| - x_3 \leq \sqrt{1 - \max\{0, x_2\}^2} \right\}$$

Final Remarks

Introduce and study the properties of $\mathcal{K}$-minimal and $\mathcal{K}$-sublinear inequalities for conic MIPs

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For two-term disjunctions on Lorentz cone, $\mathcal{L}^n$

- Derive explicit expressions for disjunctive conic cuts
  - Cover most of the recent results on conic MIR, split, and two-term disjunctive inequalities for Lorentz cones (i.e., Belotti et al.'11, Andersen & Jensen'13, Modaresi et al.'13)
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  - When we can have valid conic inequalities in the original space
  - When a single convex inequality will suffice
- Extends to disjunctions on cross-sections of the Lorentz cone (Joint work with S. Yıldız and G. Cornuéjols)

Thank you!

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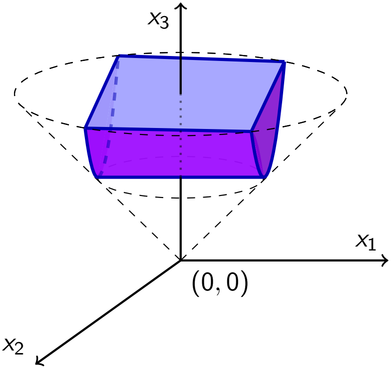
  http://arxiv.org/pdf/1404.7813v1.pdf

  A shorter version is published as part of *Proceedings of* 17^{th} *IPCO Conference*.

A Simple Example for Insufficiency of Cut Generating Functions

$\mathcal{K} = \mathcal{L}^3$, $A = [1, 0, 0]$ and $\mathcal{B} = \{-1, 1\}$, i.e.,

$\mathcal{S}(A, \mathcal{K}, \mathcal{B}) = \{x \in \mathbb{R}^3 : \ x_1 \in \{-1, 1\}, \ x_3 \geq \sqrt{x_1^2 + x_2^2}\}$



$\mathrm{conv}(\mathcal{S}(A, \mathcal{K}, \mathcal{B})) = \{x \in \mathbb{R}^3 : \ -1 \leq x_1 \leq 1, \ x_3 \geq \sqrt{1 + x_2^2}\}$

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$\mathcal{K}$-minimal inequalities are:

(a) $\mu^{(+)} = (1; 0; 0)$ with $\eta_0^{(+)} = -1$ and $\mu^{(-)} = (-1; 0; 0)$ with $\eta_0^{(-)} = -1$;

(b) $\mu^{(t)} = (0; t; \sqrt{t^2 + 1})$ with $\eta_0^{(t)} = 1$ for all $t \in \mathbb{R}$.

$\mathcal{K} = \mathcal{L}^3$, $A = [1, 0, 0]$ and $\mathcal{B} = \{-1, 1\}$, i.e.,

$$\mathcal{S}(A, \mathcal{K}, \mathcal{B}) = \{x \in \mathbb{R}^3 : \ x_1 \in \{-1, 1\}, \ x_3 \geq \sqrt{x_1^2 + x_2^2}\}$$

$$\mathrm{conv}(\mathcal{S}(A, \mathcal{K}, \mathcal{B})) = \{x \in \mathbb{R}^3 : \ -1 \leq x_1 \leq 1, \ x_3 \geq \sqrt{1 + x_2^2}\}$$

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Linear inequalities in (b) cannot be generated by any cut generating function $\rho(\cdot)$, i.e., $\rho(A^i) = \mu_i^{(t)}$ is not possible for any function $\rho(\cdot)$.

[Sharp contrast to the strong conic IP dual result of Moran, Dey, & Vielma '12.]