Structure in Mixed Integer Conic Optimization: From Minimal Inequalities to Conic Disjunctive Cuts

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Outline

- Mixed integer conic optimization (MICP)
  - Problem setting
  - Motivation

- Structure of linear valid inequalities
  - $\mathcal{K}$-minimal valid inequalities
  - $\mathcal{K}$-sublinear valid inequalities

- Disjunctive cuts for Lorentz cone (joint work with Sercan Yıldız (CMU))
  - Structure of valid linear inequalities
  - A class of valid convex inequalities
  - Nice case and nasty case (with examples)
Problem:
Study the closed convex hull of

\[ S(A, K, B) = \{ x \in E : Ax \in B, x \in K \} \]

where

- \( E \) is a finite dimensional Euclidean space with inner product \( \langle \cdot, \cdot \rangle \)
- \( A \) is a linear map from \( E \) to \( \mathbb{R}^m \)
- \( B \subset \mathbb{R}^m \) is a given set of points (can be finite or infinite)
- \( K \subset E \) is a full-dimensional, closed, convex and pointed cone [regular cone]
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- \( A \) is a linear map from \( E \) to \( \mathbb{R}^m \)
- \( \mathcal{B} \subset \mathbb{R}^m \) is a given set of points (can be finite or infinite)
- \( \mathcal{K} \subset E \) is a full-dimensional, closed, convex and pointed cone [regular cone]

Most (if not all!) cutting planes (convexification techniques) in MILPs rely on disjunctive formulations.
Representation Flexibility

\[ S(A, \mathcal{K}, \mathcal{B}) = \{ x \in E : Ax \in \mathcal{B}, \ x \in \mathcal{K} \} \]

This set captures the essential structure of MICPs
Representation Flexibility

\[ S(A, \mathcal{K}, \mathcal{B}) = \{ x \in E : Ax \in \mathcal{B}, \ x \in \mathcal{K} \} \]

This set captures the essential structure of MICPs

\[ \{(y, v) \in \mathbb{R}_+^k \times \mathbb{Z}^q : W_y + H_v - b \in \mathcal{K}\} \]

where \( \mathcal{K} \subset E \) is a full-dimensional, closed, convex, pointed cone.
\[ S(A, \mathcal{K}, B) = \{ x \in E : Ax \in B, \ x \in \mathcal{K} \} \]

This set captures the essential structure of MICPs

\[ \{ (y, v) \in \mathbb{R}^k_+ \times \mathbb{Z}^q : W y + H v - b \in \mathcal{K} \} \]

where \( \mathcal{K} \subset E \) is a full-dimensional, closed, convex, pointed cone.

Define

\[ x = \begin{pmatrix} y \\ z \end{pmatrix}, \quad A = \begin{bmatrix} W, & -\text{Id} \end{bmatrix}, \quad \text{and} \quad B = b - H \mathbb{Z}^q, \]

where \( \text{Id} \) is the identity map in \( E \). Then we arrive at

\[ S(A, \mathcal{K}', B) = \{ x \in (\mathbb{R}^k_+ \times E) : Ax \in B, \ x \in (\mathbb{R}_+^k \times \mathcal{K}) \} \]

\[ := \mathcal{K}' \]
\[ S(A, \mathcal{K}, \mathcal{B}) = \{ x \in E : Ax \in \mathcal{B}, \ x \in \mathcal{K} \} \]

- Models traditional disjunctive formulations
- Captures the essential non-convex structure of MICPs
- Precisely contains famous Gomory’s Corner Polyhedron (1969)
- Arises in modeling complementarity relations
- Can be used as a natural relaxation for sequential convexification
- etc., etc...
Questions of Interest

\[ S(A, \mathcal{K}, \mathcal{B}) = \{ x \in E : Ax \in \mathcal{B}, x \in \mathcal{K} \} \]

General Questions:

- When can we study/characterize \( \text{conv}(S(A, \mathcal{K}, \mathcal{B})) \) explicitly?
- Will (or when will) \( \text{conv}(S(A, \mathcal{K}, \mathcal{B})) \) preserve the nice structural properties we originally had?
Questions of Interest

\[ S(A, K, B) = \left\{ x \in E : Ax \in B, x \in K \right\} \]

General Questions:

- When can we study/characterize $\overline{\text{conv}}(S(A, K, B))$ explicitly?
- Will (or when will) $\overline{\text{conv}}(S(A, K, B))$ preserve the nice structural properties we originally had?

- Let’s first see what we can find out about the structure of linear valid inequalities...
Structure of Valid Linear Inequalities

\[ S(A, \mathcal{K}, \mathcal{B}) = \{ x \in E : Ax \in \mathcal{B}, \ x \in \mathcal{K} \} \]

\[ \text{\texttt{conv}}(S(A, \mathcal{K}, \mathcal{B})) = \text{intersection of all linear valid inequalities (v.i.)} \]

\[ \langle \mu, x \rangle \geq \eta_0 \text{ for } S(A, \mathcal{K}, \mathcal{B}) \]
Structure of Valid Linear Inequalities

\[ S(A, \mathcal{K}, B) = \{ x \in E : Ax \in B, \ x \in \mathcal{K} \} \]

- \( \text{conv}(S(A, \mathcal{K}, B)) = \) intersection of all linear valid inequalities (v.i.)
  \[ \langle \mu, x \rangle \geq \eta_0 \] for \( S(A, \mathcal{K}, B) \)

\[ C(A, \mathcal{K}, B) = \text{convex cone of all linear valid inequalities for } S(A, \mathcal{K}, B) \]
\[ = \left\{ (\mu; \eta_0) : \mu \in E, \mu \neq 0, -\infty < \eta_0 \leq \inf_{x \in S(A, \mathcal{K}, B)} \langle \mu, x \rangle \right\} \]

**Goal:** Study \( C(A, \mathcal{K}, B) \) in order to characterize the properties of the linear v.i., and identify the necessary and/or sufficient ones defining \( \text{conv}(S(A, \mathcal{K}, B)) \)
Which Inequalities Should We Really Care About?

\[ S(A, \mathcal{K}, \mathcal{B}) = \{ x \in E : Ax \in \mathcal{B}, \ x \in \mathcal{K} \} \]

\[ C(A, \mathcal{K}, \mathcal{B}) = \text{convex cone of } (\mu; \eta_0) \text{ of all linear valid inequalities } \langle \mu, x \rangle \geq \eta_0 \]
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\[ C(A, \mathcal{K}, \mathcal{B}) = \text{convex cone of } (\mu; \eta_0) \text{ of all linear valid inequalities } \langle \mu, x \rangle \geq \eta_0 \]

\( \Rightarrow \) Cone implied inequality, (\(\delta; 0\)) for any \(\delta \in \mathcal{K}^* \setminus \{0\}\), is always valid.
Which Inequalities Should We Really Care About?

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\[ C(A, \mathcal{K}, \mathcal{B}) = \text{convex cone of } (\mu; \eta_0) \text{ of all linear valid inequalities } \langle \mu, x \rangle \geq \eta_0 \]

**Definition**

An inequality \((\mu; \eta_0) \in C(A, \mathcal{K}, \mathcal{B})\) is a **\(\mathcal{K}\)-minimal valid inequality** if for all \(\rho\) such that \(\rho \preceq_{\mathcal{K}^*} \mu\) and \(\rho \neq \mu\), we have \(\rho_0 < \eta_0\).
Which Inequalities Should We Really Care About?

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**Definition**

An inequality \((\mu; \eta_0) \in C(A, \mathcal{K}, \mathcal{B})\) is a \(\mathcal{K}\)-minimal valid inequality if for all \(\rho\) such that \(\rho \preceq_{K^*} \mu\) and \(\rho \neq \mu\), we have \(\rho_0 < \eta_0\).

\[ C_m(A, \mathcal{K}, \mathcal{B}) = \text{cone of } \mathcal{K}\text{-minimal valid inequalities} \]
Which Inequalities Should We Really Care About?

$$S(A, K, B) = \{x \in E : Ax \in B, \ x \in K\}$$

$$C(A, K, B) = \text{convex cone of } (\mu; \eta_0) \text{ of all linear valid inequalities } \langle \mu, x \rangle \geq \eta_0$$

$$C_m(A, K, B) = \text{cone of } K\text{-minimal valid inequalities}$$
\( \mathcal{K}\)-minimal Inequalities

**Definition**

An inequality \((\mu; \eta_0) \in C(A, \mathcal{K}, B)\) is a \( \mathcal{K}\)-minimal valid inequality if for all \( \rho \) such that \( \rho \preceq_{\mathcal{K}^*} \mu \) and \( \rho \neq \mu \), we have \( \rho_0 < \eta_0 \).

\[ \Rightarrow \] A \( \mathcal{K}\)-minimal v.i. \((\mu; \eta_0)\) cannot be written as a sum of a cone implied inequality and another valid inequality.
\textbf{\(\mathcal{K}\)-minimal Inequalities}

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An inequality \((\mu; \eta_0) \in C(A, \mathcal{K}, B)\) is a \(\mathcal{K}\)-minimal valid inequality if for all \(\rho\) such that \(\rho \preceq_{\mathcal{K}^*} \mu\) and \(\rho \neq \mu\), we have \(\rho_0 < \eta_0\).

\(\Rightarrow\) A \(\mathcal{K}\)-minimal v.i. \((\mu; \eta_0)\) cannot be written as a \textit{sum} of a cone implied inequality and another valid inequality.

\(\Rightarrow\) Cone implied inequality \((\delta; 0)\) for any \(\delta \in \mathcal{K}^* \setminus \{0\}\) is \textit{never} minimal.
\( \mathcal{K} \)-minimal Inequalities

Definition

An inequality \((\mu; \eta_0) \in C(A, \mathcal{K}, B)\) is a \( \mathcal{K} \)-minimal valid inequality if for all \( \rho \) such that \( \rho \preceq_{\mathcal{K}^*} \mu \) and \( \rho \neq \mu \), we have \( \rho_0 < \eta_0 \).

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\[ \Rightarrow \] Cone implied inequality \((\delta; 0)\) for any \( \delta \in \mathcal{K}^* \setminus \{0\} \) is never minimal.

\[ \Rightarrow \] \( \mathcal{K} \)-minimal v.i. exists if and only if \( \nexists \) valid equations of form \( \langle \delta, x \rangle = 0 \) with \( \delta \in \mathcal{K}^* \setminus \{0\} \).
**K-minimal Inequalities**

**Definition**

An inequality \((\mu; \eta_0) \in C(A, K, B)\) is a **K-minimal valid inequality** if for all \(\rho\) such that \(\rho \preceq_{K^*} \mu\) and \(\rho \neq \mu\), we have \(\rho_0 < \eta_0\).

\[\Rightarrow\] A **K-minimal v.i.** \((\mu; \eta_0)\) cannot be written as a sum of a cone implied inequality and another valid inequality.

\[\Rightarrow\] Cone implied inequality \((\delta; 0)\) for any \(\delta \in K^* \setminus \{0\}\) is never minimal.

\[\Rightarrow\] **K-minimal v.i.** exists if and only if \(\not\exists\) valid equations of form \(\langle \delta, x \rangle = 0\) with \(\delta \in K^* \setminus \{0\}\).

**Theorem: [Sufficiency of K-minimal Inequalities]**

Whenever \(C_m(A, K, B) \neq \emptyset\), **K-minimal v.i.** together with \(x \in K\) constraint are sufficient to describe \(\text{conv}(S(A, K, B))\).
Some Properties

We can show that

- Every valid inequality \((\mu; \eta_0) \in C(A, \mathcal{K}, B)\) satisfies condition
  \[(A.0) \quad \mu \in \text{Im}(A^*) + \mathcal{K}^*.
\]
Some Properties

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- Every valid inequality $(\mu; \eta_0) \in C(A, \mathcal{K}, \mathcal{B})$ satisfies condition
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Remark

For any $\mu \in \text{Im}(A^*) + \mathcal{K}^*$,

\[D_\mu := \{ \lambda \in \mathbb{R}^m : \mu - A^* \lambda \in \mathcal{K}^* \} \neq \emptyset; \text{ and} \]

\[\Rightarrow \quad \text{for any } \eta_0 \leq \inf_{b \in \mathcal{B}} \sigma_{D_\mu}(b), \text{ where } \sigma_{D_\mu} := \text{support function of } D_\mu, \]

we have $(\mu; \eta_0) \in C(A, \mathcal{K}, \mathcal{B})$. 

F. Kılınç-Karzan (CMU)
Some Properties

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- Every valid inequality \((\mu; \eta_0) \in C(A, \mathcal{K}, B)\) satisfies condition
  \[
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  \]

- \(\mathcal{K}\)-minimal inequalities have more structure, i.e., they are \(\mathcal{K}\)-sublinear:

Definition

An inequality \((\mu; \eta_0)\) is a \(\mathcal{K}\)-sublinear v.i. \((C_\alpha(A, \mathcal{K}, B))\) if it satisfies

\[
(A.1) \quad 0 \leq \langle \mu, u \rangle \quad \text{for all } u \text{ s.t. } Au = 0 \text{ and } \\
\langle \alpha, v \rangle u + v \in \mathcal{K} \quad \forall v \in \text{Ext}(\mathcal{K}) \quad \text{holds for some } \alpha \in \text{Ext}(\mathcal{K}^*),
\]

\[
(A.2) \quad \eta_0 \leq \langle \mu, x \rangle \quad \text{for all } x \in S(A, \mathcal{K}, B).
\]

Condition \((A.1)\) implies \((A.0)\).
We can establish necessary, and also sufficient conditions for an inequality $(\mu; \eta_0)$ to be $\mathcal{K}$-minimal or $\mathcal{K}$-sublinear via its relation with support function $\sigma_{D_\mu}$ of the structured set $D_\mu$. 

This recovers a number of results from Johnson'81, and Conforti et al.'13. This is underlies a cut generating function view for MILPs.
On Conditions for $\mathcal{K}$-minimality and $\mathcal{K}$-sublinearity

- We can establish necessary, and also sufficient conditions for an inequality $(\mu; \eta_0)$ to be $\mathcal{K}$-minimal or $\mathcal{K}$-sublinear via its relation with support function $\sigma_{D\mu}$ of the structured set $D\mu$.

- When $\mathcal{K} = \mathbb{R}_+^n$:
  - $\mathcal{K}$-sublinear v.i. are identical to the class of subadditive v.i. defined by Johnson’81, i.e., condition (A.1) is precisely condition (A.0) and
    \[(A.1i) \quad \text{for all } i = 1, \ldots, n, \quad \mu_i \leq \langle \mu, u \rangle \text{ for all } u \in \mathbb{R}_+^n \text{ s.t. } Au = A^i.\]
On Conditions for $\mathcal{K}$-minimality and $\mathcal{K}$-sublinearity

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  - Our sufficient condition for $\mathcal{K}$-sublinearity matches precisely our necessary condition, resulting in
    \[(\mu; \eta_0) \in C_a(A, \mathcal{K}, B) \iff \mu_i = \sigma_{D_\mu}(A^i) \text{ for all } i.\]

  \[\Rightarrow \text{ All } \mathbb{R}_+^n\text{-sublinear (and thus } \mathbb{R}_+^n\text{-minimal) inequalities are generated by sublinear functions (subadditive and positively homogeneous, in fact also piecewise linear and convex), i.e., support functions } \sigma_{D_\mu}(\cdot) \text{ of } D_\mu.\]

This recovers a number of results from Johnson’81, and Conforti et al.’13.
On Conditions for $\mathcal{K}$-minimality and $\mathcal{K}$-sublinearity

- We can establish necessary, and also sufficient conditions for an inequality $(\mu; \eta_0)$ to be $\mathcal{K}$-minimal or $\mathcal{K}$-sublinear via its relation with support function $\sigma_{D\mu}$ of the structured set $D\mu$.
- When $\mathcal{K} = \mathbb{R}_+^n$:
  - $\mathcal{K}$-sublinear v.i. are identical to the class of subadditive v.i. defined by Johnson’81, i.e., condition (A.1) is precisely condition (A.0) and
    \begin{equation}
    (A.1i) \quad \text{for all } i = 1, \ldots, n, \mu_i \leq \langle \mu, u \rangle \text{ for all } u \in \mathbb{R}_+^n \text{ s.t. } Au = A^i.
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    \[(\mu; \eta_0) \in C_a(A, \mathcal{K}, B) \iff \mu_i = \sigma_{D\mu}(A^i) \text{ for all } i.\]

$\Rightarrow$ All $\mathbb{R}_+^n$-sublinear (and thus $\mathbb{R}_+^n$-minimal) inequalities are generated by sublinear functions (subadditive and positively homogeneous, in fact also piecewise linear and convex), i.e., support functions $\sigma_{D\mu}(\cdot)$ of $D\mu$.

[This recovers a number of results from Johnson’81, and Conforti et al.’13.]

$\Rightarrow$ This is underlies a cut generating function view for MILPs.
For general regular cones $\mathcal{K}$ other than $\mathbb{R}^n_+$, unfortunately there is a gap between our current necessary condition and our sufficient condition for $\mathcal{K}$-minimality.

Moreover, there is a simple example $S(A, \mathcal{K}, \mathcal{B})$ with $\mathcal{K} = \mathcal{L}^3$, where a necessary (in terms of convex hull description) family of $\mathcal{K}$-minimal inequalities cannot be generated by any class of cut generating functions. This is in sharp contrast

- to MILP case, i.e., $\mathcal{K} = \mathbb{R}^n_+$, and,
- to the strong dual for MICP result of Moran et al.’12, which studied a more restrictive conic setup.
Questions on Structure

\[ S(A, K, B) = \{ x \in E : Ax \in B, \ x \in K \} \]

General Questions:

- When can we characterize \( \overline{\text{conv}}(S(A, K, B)) \) explicitly?
- Will (or when will) \( \overline{\text{conv}}(S(A, K, B)) \) preserve the nice structural properties we originally had?
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- General case is too general for us to answer these questions...
- In the rest of this talk, we will study a simple (?) yet interesting case
- Joint work with S. Yıldız
Disjunctive Cuts for Lorentz Cone, $\mathcal{L}^n$

- Start with a simple set for $x$, i.e., a regular cone $\mathcal{K} \subseteq \mathbb{R}^n$.
- Consider a **two-term disjunction**: either $c_1^T x \geq c_{1,0}$ or $c_2^T x \geq c_{2,0}$ must hold.
- Let $C_i := \{x : c_i^T x \geq c_{i,0}, \ x \in \mathcal{K}\}$.

A special case is **split disjunctions**, i.e., $c_1 = -\tau c_2$ for some $\tau > 0$, and $c_{1,0}c_{2,0} > 0$. 
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- Let $C_i := \{x : c_i^T x \geq c_{i,0}, \ x \in \mathcal{K}\}$.
- By setting

$$A = \begin{bmatrix} c_1^T \\ c_2^T \end{bmatrix}, \text{ and } B = \left\{ \begin{bmatrix} \{c_{1,0}\} + \mathbb{R}_+ \\ \mathbb{R} \end{bmatrix} \bigcup \begin{bmatrix} \{c_{2,0}\} + \mathbb{R}_+ \\ \mathbb{R} \end{bmatrix} \right\}$$

we arrive at

$$S(A, \mathcal{K}, B) = \{x \in \mathbb{R}^n : Ax \in B, \ x \in \mathcal{K}\} = C_1 \cup C_2.$$
Disjunctive Cuts for Lorentz Cone, $\mathcal{L}^n$

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- Consider a two-term disjunction: either $c_1^T x \geq c_{1,0}$ or $c_2^T x \geq c_{2,0}$ must hold.
- Let $C_i := \{ x : c_i^T x \geq c_{i,0}, \ x \in \mathcal{K} \}$.
- By setting
  \[
  A = \begin{bmatrix} c_1^T \\ c_2^T \end{bmatrix}, \quad \text{and} \quad B = \left\{ \begin{bmatrix} c_{1,0} \\ \mathbb{R} \end{bmatrix} + \mathbb{R}_+ \right\} \cup \left\{ \begin{bmatrix} \mathbb{R} \\ c_{2,0} \end{bmatrix} + \mathbb{R}_+ \right\}
  \]
  we arrive at
  \[
  S(A, \mathcal{K}, B) = \{ x \in \mathbb{R}^n : Ax \in B, \ x \in \mathcal{K} \} = C_1 \cup C_2.
  \]

We are interested in describing $\overline{\text{conv}}(S(A, \mathcal{K}, B))$ in the original space of variables:
- Is there any structure in $\overline{\text{conv}}(S(A, \mathcal{K}, B))$?
- Can we preserve the simple conic structure we started out with?
Disjunctive Cuts for Lorentz Cone, $\mathcal{L}^n$

- Start with a simple set for $x$, i.e., a regular cone $\mathcal{K} \subseteq \mathbb{R}^n$
- Consider a two-term disjunction: either $c_1^T x \geq c_{1,0}$ or $c_2^T x \geq c_{2,0}$ must hold.
- Let $C_i := \{ x : c_i^T x \geq c_{i,0}, \ x \in \mathcal{K} \}$.
- By setting $A = \begin{bmatrix} c_1^T \\ c_2^T \end{bmatrix}$, and $B = \left\{ \begin{bmatrix} \{c_{1,0}\} + \mathbb{R}_+ \\ \mathbb{R} \end{bmatrix} \cup \begin{bmatrix} \{c_{2,0}\} + \mathbb{R}_+ \\ \mathbb{R} \end{bmatrix} \right\}$

we arrive at

$$S(A, \mathcal{K}, B) = \{ x \in \mathbb{R}^n : Ax \in B, \ x \in \mathcal{K} \} = C_1 \cup C_2.$$

$\Rightarrow$ Simple set we start out can be $\mathcal{U} = \{ x \in \mathbb{R}^n : Qx - d \in \mathcal{K} \}$ with $Q \in \mathbb{R}^{m \times n}$ having full row rank.
Approach

- Characterize the structure of $\mathcal{K}$-minimal and tight valid linear inequalities
Approach

- Characterize the structure of $\mathcal{K}$-minimal and tight valid linear inequalities
- Using conic duality, we group these linear inequalities appropriately, and thus derive a family of convex valid inequalities sufficient to describe the closed convex hull
Characterize the structure of $\mathcal{K}$-minimal and tight valid linear inequalities

Using conic duality, we group these linear inequalities appropriately, and thus derive a family of convex valid inequalities sufficient to describe the closed convex hull

Any structure beyond convexity?

⇒ Understand when these convex inequalities are conic representable
Approach

- Characterize the structure of \( K \)-minimal and tight valid linear inequalities

- Using conic duality, we group these linear inequalities appropriately, and thus derive a family of convex valid inequalities sufficient to describe the closed convex hull

- Any structure beyond convexity?
  - ⇒ Understand when these convex inequalities are conic representable

- When does a single inequality from this family suffice?
  - ⇒ Characterize when only single inequality from this family is sufficient to describe the closed convex hull
Setup for Conic Disjunctive Cuts

- Start with a simple set for $x$, i.e., $K$
- Consider a two-term disjunction: either $c_1^T x \geq c_{1,0}$ or $c_2^T x \geq c_{2,0}$ must hold.
- Let $C_i := \{x \in K : c_i^T x \geq c_{i,0}\}$. 

WLOG we assume that $c_{1,0}, c_{2,0} \in \{0, \pm 1\}$ and $C_1 \neq \emptyset$ and $C_2 \neq \emptyset$ (in fact we assume $C_1, C_2$ are strictly feasible) $C_1 \neq C_2$ and $C_2 \neq C_1$.

This assumption is equivalent to Assumption 5.10: The disjunction $c_1^T x \geq c_{1,0}$ or $c_2^T x \geq c_{2,0}$ satisfies

$$\{\beta \in \mathbb{R}_+ : \beta c_{1,0} \geq c_{2,0}, \beta c_{2,0} \geq c_{1,0}\} = \emptyset,$$

and

$$\{\beta \in \mathbb{R}_+ : \beta c_{2,0} \geq c_{1,0}, \beta c_{1,0} \geq c_{2,0}\} = \emptyset.$$
Setup for Conic Disjunctive Cuts

- Start with a simple set for $x$, i.e., $\mathcal{K}$
- Consider a two-term disjunction: either $c_1^T x \geq c_{1,0}$ or $c_2^T x \geq c_{2,0}$ must hold.
- Let $C_i := \{x \in \mathcal{K} : c_i^T x \geq c_{i,0}\}$.
- WLOG we assume that
  - $c_{1,0}, c_{2,0} \in \{0, \pm 1\}$ and
  - $C_1 \neq \emptyset$ and $C_2 \neq \emptyset$ (in fact we assume $C_1, C_2$ are strictly feasible)
  - $C_1 \not\subseteq C_2$ and $C_2 \not\subseteq C_1$. 
Setup for Conic Disjunctive Cuts

- Start with a simple set for $x$, i.e., $\mathcal{K}$
- Consider a two-term disjunction: either $c_1^T x \geq c_{1,0}$ or $c_2^T x \geq c_{2,0}$ must hold.
- Let $C_i := \{ x \in \mathcal{K} : c_i^T x \geq c_{i,0} \}$.
- WLOG we assume that
  - $c_{1,0}, c_{2,0} \in \{0, \pm 1\}$ and
  - $C_1 \neq \emptyset$ and $C_2 \neq \emptyset$ (in fact we assume $C_1, C_2$ are strictly feasible)
  - $C_1 \not\subseteq C_2$ and $C_2 \not\subseteq C_1$. This assumption is equivalent to

Assumption

The disjunction $c_1^T x \geq c_{1,0}$ or $c_2^T x \geq c_{2,0}$ satisfies

- $\{ \beta \in \mathbb{R}_+ : \beta c_{1,0} \geq c_{2,0}, \ c_2 - \beta c_1 \in \mathcal{K} \} = \emptyset$, and
- $\{ \beta \in \mathbb{R}_+ : \beta c_{2,0} \geq c_{1,0}, \ c_1 - \beta c_2 \in \mathcal{K} \} = \emptyset$. 
Disjunction on $\mathcal{K}$: either $c_1^T x \geq c_{1,0}$ or $c_2^T x \geq c_{2,0}$ with $c_{1,0}, c_{2,0} \in \{0, \pm 1\}$

**Standard Approach**

For any valid linear inequality, $\mu^T x \geq \mu_0$ for $\text{conv}(C_1 \cup C_2)$ there exists $\alpha_1, \alpha_2 \in \mathcal{K}$, and $\beta_1, \beta_2 \in \mathbb{R}^+$ s.t.

\[
\begin{align*}
\mu &= \alpha_1 + \beta_1 c_1, \\
\mu &= \alpha_2 + \beta_2 c_2, \\
\mu_0 &\leq \min \{\beta_1 c_{1,0}, \beta_2 c_{2,0}\}.
\end{align*}
\]
Disjunction on $\mathcal{K}$: either $c_1^T x \geq c_{1,0}$ or $c_2^T x \geq c_{2,0}$ with $c_{1,0}, c_{2,0} \in \{0, \pm 1\}$

**Proposition**

For any $\mathcal{K}$-minimal and tight valid linear inequality, $\mu^T x \geq \mu_0$ for $\text{conv}(C_1 \cup C_2)$ there exists $\alpha_1, \alpha_2 \in \text{bd}(\mathcal{K})$, and $\beta_1, \beta_2 \in (\mathbb{R}_+ \setminus \{0\})$ s.t.

$$
\mu = \alpha_1 + \beta_1 c_1, \\
\mu = \alpha_2 + \beta_2 c_2,
$$

$$
\min\{c_{1,0}\beta_1, c_{2,0}\beta_2\} = \mu_0 = \min\{c_{1,0}, c_{2,0}\},
$$

and at least one of $\beta_1$ and $\beta_2$ is equal to 1.
Assume $c_{1,0} \geq c_{2,0} \Rightarrow \beta_2 = 1$
Deriving a Nonlinear (but Convex) Valid Inequality when $\mathcal{K} = \mathcal{L}^n$

Assume $c_{1,0} \geq c_{2,0} \implies \beta_2 = 1$

Consider the set of undominated v.i. $\mu^T x \geq \mu_0$ for given $\beta_1 = \beta > 0$ and $\beta_2 = 1$, i.e., $\mu_0 = \min\{c_{1,0}, c_{2,0}\} = c_{2,0}$ and

$$\mu \in \mathcal{M}(\beta, 1) := \{\mu \in \mathbb{R}^n : \exists \alpha_1, \alpha_2 \in \text{bd} \mathcal{L}^n \text{ s.t. } \mu = \alpha_1 + \beta c_1 = \alpha_2 + c_2\}$$
Assume \( c_{1,0} \geq c_{2,0} \) \( \Rightarrow \) \( \beta_2 = 1 \)

Consider the set of undominated v.i. \( \mu^T x \geq \mu_0 \) for given \( \beta_1 = \beta > 0 \) and \( \beta_2 = 1 \), i.e., \( \mu_0 = \min \{ c_{1,0}, c_{2,0} \} = c_{2,0} \) and

\[
\mu \in M(\beta, 1) := \{ \mu \in \mathbb{R}^n : \exists \alpha_1, \alpha_2 \in \text{bd } \mathcal{L}^n \text{ s.t. } \mu = \alpha_1 + \beta c_1 = \alpha_2 + c_2 \}
\]

\[
= \left\{ \mu \in \mathbb{R}^n : \tilde{\mu}^T (\beta \tilde{c}_1 - \tilde{c}_2) - \mu_n (\beta c_{1,n} - c_{2,n}) = \frac{M}{2}, \| \mu - \beta \tilde{c}_1 \|_2 = \mu_n - \beta c_{1,n} \right\}
\]

where \( M := (\beta^2 \| \tilde{c}_1 \|_2^2 - \| \tilde{c}_2 \|_2^2) - (\beta^2 c_{1,n}^2 - c_{2,n}^2) \).
Deriving a Nonlinear (but Convex) Valid Inequality when $K = L^n$

Assume $c_{1,0} \geq c_{2,0} \Rightarrow \beta_2 = 1$

Consider the set of undominated v.i. $\mu^T x \geq \mu_0$ for given $\beta_1 = \beta > 0$ and $\beta_2 = 1$,
i.e., $\mu_0 = \min\{c_{1,0}, c_{2,0}\} = c_{2,0}$ and

$$\mu \in M(\beta, 1) := \{\mu \in \mathbb{R}^n : \exists \alpha_1, \alpha_2 \in \text{bd } L^n \text{ s.t. } \mu = \alpha_1 + \beta c_1 = \alpha_2 + c_2\}$$

$$= \left\{\mu \in \mathbb{R}^n : \tilde{\mu}^T (\beta \tilde{c}_1 - \tilde{c}_2) - \mu_n (\beta c_{1,n} - c_{2,n}) = \frac{M}{2}, \|\tilde{\mu} - \beta \tilde{c}_1\|_2 = \mu_n - \beta c_{1,n}\right\}$$

where $M := (\beta^2 \|\tilde{c}_1\|_2^2 - \|\tilde{c}_2\|_2^2) - (\beta^2 c_{1,n}^2 - c_{2,n}^2)$.

$x \in \text{conv}(C_1 \cup C_2) \Rightarrow x \in L^n$ and $\mu^T x \geq c_{2,0}$ $\forall \mu \in M(\beta, 1)$
Deriving a Nonlinear (but Convex) Valid Inequality when $\mathcal{K} = \mathcal{L}^n$

Assume $c_{1,0} \geq c_{2,0} \Rightarrow \beta_2 = 1$

Consider the set of undominated v.i. $\mu^T x \geq \mu_0$ for given $\beta_1 = \beta > 0$ and $\beta_2 = 1$,

i.e., $\mu_0 = \min\{c_{1,0}, c_{2,0}\} = c_{2,0}$ and

$$\mu \in \mathcal{M}(\beta, 1) := \{\mu \in \mathbb{R}^n : \exists \alpha_1, \alpha_2 \in \text{bd } \mathcal{L}^n \text{ s.t. } \mu = \alpha_1 + \beta c_1 = \alpha_2 + c_2\}$$

$$= \left\{\mu \in \mathbb{R}^n : \mu^T (\beta \tilde{c}_1 - \tilde{c}_2) - \mu_n (\beta c_{1,n} - c_{2,n}) = \frac{M}{2}, \|\mu - \beta \tilde{c}_1\|_2 = \mu_n - \beta c_{1,n}\right\}$$

where $M := (\beta^2 \|\tilde{c}_1\|_2^2 - \|\tilde{c}_2\|_2^2) - (\beta^2 c_{1,n}^2 - c_{2,n}^2)$.

$$x \in \text{conv}(C_1 \cup C_2) \Rightarrow x \in \mathcal{L}^n \text{ and } \mu^T x \geq c_{2,0} \forall \mu \in \mathcal{M}(\beta, 1)$$

$$\Leftrightarrow x \in \mathcal{L}^n \text{ and } \inf_{\mu} \{\mu^T x : \mu \in \mathcal{M}(\beta, 1)\} \geq c_{2,0}$$
Assume $c_{1,0} \geq c_{2,0} \implies \beta_2 = 1$

Consider the set of undominated v.i. $\mu^T x \geq \mu_0$ for given $\beta_1 = \beta > 0$ and $\beta_2 = 1$, i.e., $\mu_0 = \min\{c_{1,0}, c_{2,0}\} = c_{2,0}$ and

$$\mu \in \mathcal{M}(\beta, 1):= \{\mu \in \mathbb{R}^n : \exists \alpha_1, \alpha_2 \in \text{bd} \mathcal{L}^n \text{ s.t. } \mu = \alpha_1 + \beta c_1 = \alpha_2 + c_2\}$$

$$= \left\{\mu \in \mathbb{R}^n : \tilde{\mu}^T (\beta \tilde{c}_1 - \tilde{c}_2) - \mu_n (\beta c_{1,n} - c_{2,n}) = \frac{M}{2}, \|\tilde{\mu} - \beta \tilde{c}_1\|_2 = \mu_n - \beta c_{1,n}\right\}$$

where $M := (\beta^2 \|\tilde{c}_1\|_2^2 - \|\tilde{c}_2\|_2^2) - (\beta^2 c_{1,n}^2 - c_{2,n}^2)$.

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$\iff x \in \mathcal{L}^n$ and $\inf_{\mu} \{\mu^T x : \mu \in \mathcal{M}(\beta, 1)\} \geq c_{2,0}$

$\iff x \in \mathcal{L}^n$ and $\inf_{\mu} \{\mu^T x : \mu \in \overline{\mathcal{M}(\beta, 1)}\} \geq c_{2,0}$
Assume $c_{1,0} \geq c_{2,0} \implies \beta_2 = 1$

Consider the set of undominated v.i. $\mu^T x \geq \mu_0$ for given $\beta_1 = \beta > 0$ and $\beta_2 = 1$, i.e., $\mu_0 = \min \{ c_{1,0}, c_{2,0} \} = c_{2,0}$ and

$$
\mu \in M(\beta, 1) := \{ \mu \in \mathbb{R}^n : \exists \alpha_1, \alpha_2 \in \text{bd} \mathcal{L}^n \text{ s.t. } \mu = \alpha_1 + \beta c_1 = \alpha_2 + c_2 \}
$$

$$=
\left\{ \mu \in \mathbb{R}^n : \tilde{\mu}^T (\beta \tilde{c}_1 - \tilde{c}_2) - \mu_n (\beta c_{1,n} - c_{2,n}) = \frac{M}{2}, \| \tilde{\mu} - \beta \tilde{c}_1 \|_2 = \mu_n - \beta c_{1,n} \right\}
$$

where $M := (\beta^2 \| \tilde{c}_1 \|_2^2 - \| \tilde{c}_2 \|_2^2) - (\beta^2 c_{1,n}^2 - c_{2,n}^2)$.

$$x \in \overline{\text{conv}}(C_1 \cup C_2) \implies x \in \mathcal{L}^n \text{ and } \mu^T x \geq c_{2,0} \quad \forall \mu \in M(\beta, 1)
$$

$$\iff x \in \mathcal{L}^n \text{ and } \inf_{\mu} \{ \mu^T x : \mu \in M(\beta, 1) \} \geq c_{2,0}
$$

$$\iff x \in \mathcal{L}^n \text{ and } \inf_{\mu} \{ \mu^T x : \mu \in \widehat{M}(\beta, 1) \} \geq c_{2,0}
$$

$$\iff x \in \mathcal{L}^n \text{ and } \max_{\rho, \tau} \left\{ \beta c_1^T \rho + \frac{M}{2} \tau : \rho + \tau \left( \begin{array}{c} \beta \tilde{c}_1 - \tilde{c}_2 \\ -\beta c_{1,n} + c_{2,n} \end{array} \right) = x, \rho \in \mathcal{L}^n \right\} \geq c_{2,0}
$$
Convex Disjunctive Cut for $\mathcal{K} = \mathcal{L}^n$

Disjunction on second-order cone, $\mathcal{L}^n = \{ x \in \mathbb{R}^n : x_n \geq \|\tilde{x}\|_2 \}$: either $c_1^T x \geq c_{1,0}$ or $c_2^T x \geq c_{2,0}$ with $c_{1,0} \geq c_{2,0}$

**Theorem**

For any $\beta > 0$ s.t. $\beta c_{1,0} \geq c_{2,0}$ and $\beta c_1 - c_2 \notin \pm \text{int}(\mathcal{L}^n)$, then the following inequality is valid for $\overline{\text{conv}}(C_1 \cup C_2)$:

$$2c_{2,0} - (\beta c_1 + c_2)^T x \leq \sqrt{((\beta c_1 - c_2)^T x)^2 + N(\beta) \times (x_n^2 - \|\tilde{x}\|_2^2)}$$

where $N(\beta) := \|\beta \tilde{c}_1 - \tilde{c}_2\|_2^2 - (\beta c_{1,n} - c_{2,n})^2$. 
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$$2c_{2,0} - (\beta c_1 + c_2)^T x \leq \sqrt{((\beta c_1 - c_2)^T x)^2 + N(\beta) \cdot (x_n^2 - \|\tilde{x}\|_2^2)}$$

where $N(\beta) := \|\beta \tilde{c}_1 - \tilde{c}_2\|_2^2 - (\beta c_{1,n} - c_{2,n})^2$.

From its construction, this inequality

- exactly captures all undominated linear v.i. $\mu^T x \geq \mu_0$ corresponding to $\beta_1 = \beta$ and $\beta_2 = 1$, e.g., $\mu_0 = c_{2,0}$ and $\mu \in \mathcal{M}(\beta, 1)$
Convex Disjunctive Cut for $\mathcal{K} = \mathcal{L}^n$

Disjunction on second-order cone, $\mathcal{L}^n = \{ x \in \mathbb{R}^n : x_n \geq \|\tilde{x}\|_2 \}$: either $c_1^T x \geq c_{1,0}$ or $c_2^T x \geq c_{2,0}$ with $c_{1,0} \geq c_{2,0}$

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- is valid and convex
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Theorem

For any $\beta > 0$ s.t. $\beta c_{1,0} \geq c_{2,0}$ and $\beta c_1 - c_2 \notin \pm \text{int}(\mathcal{L}^n)$, then the following inequality is valid for $\text{conv}(C_1 \cup C_2)$:

$$2c_{2,0} - (\beta c_1 + c_2)^\top x \leq \sqrt{((\beta c_1 - c_2)^\top x)^2 + N(\beta) \times (x_n^2 - \|\tilde{x}\|_2^2)}$$

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From its construction, this inequality

- exactly captures all undominated linear v.i. $\mu^T x \geq \mu_0$ corresponding to $\beta_1 = \beta$ and $\beta_2 = 1$, e.g., $\mu_0 = c_{2,0}$ and $\mu \in \mathcal{M}(\beta, 1)$
- is valid and convex
- reduces to the linear inequality $\beta c_1^T x \geq c_{2,0}$ in $\mathcal{L}^n$ when $\beta c_1 - c_2 \in \pm \text{bd} \mathcal{L}^n$
Structure of Convex Valid Inequalities

\[ 2c_{2,0} - (\beta c_1 + c_2)^\top x \leq \sqrt{((\beta c_1 - c_2)^\top x)^2 + N(\beta) * (x_n^2 - \|\tilde{x}\|_2^2)} \]

Any further structure than convexity?
2c_{2,0} - (\beta c_1 + c_2)^T x \leq \sqrt{(( \beta c_1 - c_2)^T x)^2 + N(\beta) * (x_n^2 - \|\tilde{x}\|^2)}

Any further structure than convexity?

Proposition

An equivalent conic quadratic form given by

\[ N(\beta)x + 2(c_2^T x - c_{2,0}) \begin{pmatrix} \beta \tilde{c}_1 - \tilde{c}_2 \\ -\beta c_1, n + c_2, n \end{pmatrix} \in \mathcal{L}^n \]

is valid whenever a symmetry condition, e.g.,

\[ -2c_{2,0} + (\beta c_1 + c_2)^T x \leq \sqrt{(( \beta c_1 - c_2)^T x)^2 + N(\beta) (x_n^2 - \|\tilde{x}\|^2)} \]

holds for all \( x \in \text{conv}(C_1 \cup C_2) \).
Structure of Convex Valid Inequalities

\[
2c_{2,0} - (\beta c_1 + c_2)\top x \leq \sqrt{((\beta c_1 - c_2)\top x)^2 + N(\beta) * (x_n^2 - ||\tilde{x}||_2^2)}
\]

Any further structure than convexity?

\[
N(\beta)x + 2(c_2\top x - c_{2,0}) \begin{pmatrix}
\beta \tilde{c}_1 - \tilde{c}_2 \\
-\beta c_{1,n} + c_{2,n}
\end{pmatrix} \in \mathcal{L}^n
\]

An equivalent conic quadratic form is valid, e.g., symmetry condition holds, when

- \( C_1 \cap C_2 = \emptyset \), i.e., a proper split disjunction, or
- \( \{x \in \mathcal{L}^n : \beta c_1\top x \geq c_{2,0}, \ c_2\top x \geq c_{2,0}\} = \{x \in \mathcal{L}^n : \beta c_1\top x = c_{2,0}, \ c_2\top x = c_{2,0}\} \)
When does a Single Convex Inequality Suffice?

A parametric family of convex inequalities:
For any $\beta > 0$ s.t. $\beta c_{1,0} \geq c_{2,0}$ and $\beta c_1 - c_2 \notin \pm \text{int}(\mathcal{L}^n)$,

$$2c_{2,0} - (\beta c_1 + c_2)^\top x \leq \sqrt{((\beta c_1 - c_2)^\top x)^2 + N(\beta) \ast (x_n^2 - \|\tilde{x}\|_2^2)}$$

is valid for $\overline{\text{conv}}(C_1 \cup C_2)$.
When does a Single Convex Inequality Suffice?

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For any $\beta > 0$ s.t. $\beta c_{1,0} \geq c_{2,0}$ and $\beta c_1 - c_2 \notin \pm \text{int}(\mathcal{L}^n)$,

$$2c_{2,0} - (\beta c_1 + c_2) \top x \leq \sqrt{((\beta c_1 - c_2) \top x)^2 + N(\beta) \ast (x_n^2 - ||\tilde{x}||_2^2)}$$

is valid for $\overline{\text{conv}}(C_1 \cup C_2)$.

Theorem
In certain cases such as

- $c_1 \in \mathcal{L}^n$ or $c_2 \in \mathcal{L}^n$, or
- $c_{1,0} = c_{2,0} \in \{\pm 1\}$ and $\text{conv}(C_1 \cup C_2)$ is closed, e.g., in the case of split disjunctions,

it is sufficient (for $\overline{\text{conv}}(C_1 \cup C_2)$) to consider only one inequality with $\beta = 1$. 
Example: Nice Case

Disjunction: \( x_3 \geq 1 \) or \( x_1 + x_3 \geq 1 \)

\[
\overline{\text{conv}}(C_1 \cup C_2) = \left\{ x \in \mathcal{L}^3 : 2 - (x_1 + 2x_3) \leq \sqrt{x_3^2 - x_2^2} \right\}
\]
Example: Nice Case

Disjunction: \[ x_3 \geq 1 \quad \text{or} \quad x_1 + x_3 \geq 1 \]

\[ \text{conv}(C_1 \cup C_2) = \left\{ x \in \mathcal{L}^3 : 2 - (x_1 + 2x_3) \leq \sqrt{x_3^2 - x_2^2} \right\} \]
There are cases when we need \textit{infinitely many} convex inequalities from this family.

- Recessive directions, and
- $\text{conv}(C_1 \cup C_2)$ being non-closed,

play a key role in these cases.
There are cases when we need infinitely many convex inequalities from this family.

- Recessive directions, and
- \( \text{conv}(C_1 \cup C_2) \) being non-closed,

play a key role in these cases.

We can still give expressions for a single inequality describing \( \text{conv}(C_1 \cup C_2) \), but it is really nasty looking...
Example: Nasty Case

Disjunction: \(-x_3 \geq -1\) or \(-x_2 \geq 0\)

\[
\text{conv}(C_1 \cup C_2) = \left\{ x \in \mathcal{L}^3 : x_2 \leq 1, 1 + |x_1| - x_3 \leq \sqrt{1 - \max\{0, x_2\}^2} \right\}
\]
Final Remarks

Introduce and study the properties of $\mathcal{K}$-minimal and $\mathcal{K}$-sublinear inequalities for conic MIPs

For two-term disjunctions on Lorentz cone,

Derive explicit expressions for disjunctive conic cuts

Cover most of the recent results on conic MIR, split, and two-term disjunctive inequalities for Lorentz cones (i.e., Belotti et al.’11, Andersen & Jensen’13, Modaresi et al.’13)

Extends to elementary split disjunctions on $p$-order cones

More intuitive and elegant derivations leading to new insights

When we can have valid conic inequalities in the original space

When a single convex inequality will suffice

Extends to disjunctions on cross-sections of the Lorentz cone (Joint work with S. Yıldız and G. Cornuëjols)
Introduce and study the properties of $\mathcal{K}$-minimal and $\mathcal{K}$-sublinear inequalities for conic MIPs

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Final Remarks

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Thank you!
http://www.optimization-online.org/DB_HTML/2013/06/3936.html

A shorter version is published as part of *Proceedings of 17th IPCO Conference*. 
A Simple Example for Insufficiency of Cut Generating Functions

\( \mathcal{K} = \mathcal{L}^3, \ A = [1, 0, 0] \) and \( B = \{-1, 1\} \), i.e.,

\[
S(A, \mathcal{K}, B) = \{x \in \mathbb{R}^3 : x_1 \in \{-1, 1\}, \ x_3 \geq \sqrt{x_1^2 + x_2^2}\}
\]

\[
\text{conv}(S(A, \mathcal{K}, B)) = \{x \in \mathbb{R}^3 : -1 \leq x_1 \leq 1, \ x_3 \geq \sqrt{1 + x_2^2}\}
\]
A Simple Example for Insufficiency of Cut Generating Functions

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\[ \text{conv}(S(A, \mathcal{K}, B)) = \{ x \in \mathbb{R}^3 : -1 \leq x_1 \leq 1, \ x_3 \geq \sqrt{1 + x_2^2} \} \]

\( \mathcal{K} \)-minimal inequalities are:

(a) \( \mu^{(+)} = (1; 0; 0) \) with \( \eta_0^{(+)} = -1 \) and \( \mu^{(-)} = (-1; 0; 0) \) with \( \eta_0^{(-)} = -1 \);

(b) \( \mu^{(t)} = (0; t; \sqrt{t^2 + 1}) \) with \( \eta_0^{(t)} = 1 \) for all \( t \in \mathbb{R} \).

[Sharp contrast to the strong conic IP dual result of Moran, Dey, & Vielma '12.]

F. Kilinc-Karzan (CMU) Structure in Mixed Integer Conic Sets
A Simple Example for Insufficiency of Cut Generating Functions

\(\mathcal{K} = \mathcal{L}^3, \ A = [1, 0, 0]\) and \(B = \{-1, 1\}\), i.e.,

\[S(A, \mathcal{K}, B) = \{x \in \mathbb{R}^3 : x_1 \in \{-1, 1\}, \ x_3 \geq \sqrt{x_1^2 + x_2^2}\}\]

\(\text{conv}(S(A, \mathcal{K}, B)) = \{x \in \mathbb{R}^3 : -1 \leq x_1 \leq 1, \ x_3 \geq \sqrt{1 + x_2^2}\}\)

**\(\mathcal{K}\)-minimal inequalities** are:

(a) \(\mu^{(+)} = (1; 0; 0)\) with \(\eta^{(+)}_0 = -1\) and \(\mu^{(-)} = (-1; 0; 0)\) with \(\eta^{(-)}_0 = -1\);  
(b) \(\mu^{(t)} = (0; t; \sqrt{t^2 + 1})\) with \(\eta^{(t)}_0 = 1\) for all \(t \in \mathbb{R}\).

(These can be expressed as a single conic inequality \(x_3 \geq \sqrt{1 + x_2^2}\).)
A Simple Example for Insufficiency of Cut Generating Functions

\( \mathcal{K} = \mathcal{L}^3, \ A = [1, 0, 0] \) and \( \mathcal{B} = \{-1, 1\} \), i.e.,

\[
S(A, \mathcal{K}, \mathcal{B}) = \{ x \in \mathbb{R}^3 : x_1 \in \{-1, 1\}, \ x_3 \geq \sqrt{x_1^2 + x_2^2} \}
\]

\[
\text{conv}(S(A, \mathcal{K}, \mathcal{B})) = \{ x \in \mathbb{R}^3 : -1 \leq x_1 \leq 1, \ x_3 \geq \sqrt{1 + x_2^2} \}
\]

\( \mathcal{K} \)-minimal inequalities are:

(a) \( \mu^{(+)} = (1; 0; 0) \) with \( \eta_0^{(+)} = -1 \) and \( \mu^{(-)} = (-1; 0; 0) \) with \( \eta_0^{(-)} = -1 \);

(b) \( \mu^{(t)} = (0; t; \sqrt{t^2 + 1}) \) with \( \eta_0^{(t)} = 1 \) for all \( t \in \mathbb{R} \).

(These can be expressed as a single conic inequality \( x_3 \geq \sqrt{1 + x_2^2} \).)

Linear inequalities in (b) cannot be generated by any cut generating function \( \rho(\cdot) \), i.e., \( \rho(A^i) = \mu_i^{(t)} \) is not possible for any function \( \rho(\cdot) \).

[Sharp contrast to the strong conic IP dual result of Moran, Dey, & Vielma '12. ]