Recent Developments in LINDO Global Solver

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Outline

- Lindo Global Solver Overview
- Recent Developments
  - Identify Convex and Concave Constraints
  - Support Semidefinite Programming (SDP)
  - Perspective Reformulation
- Concluding Remarks

Lindo Global Solver Overview

- LINDO API includes a range of solvers
  - Primal and dual simplex method - Large scale LP
  - Barrier method - Large scale LP, convex QP, SOCP and SDP
  - GRG (Generalized Reduced Gradient) and SQP method - NLP
  - Branch-and-bound and cuts IP solver works with MILP, convex MIQP and MINLP
  - Branch-and-bound global solver for non-convex NLP and MINLP

Lindo Global Solver Overview (Cont'd)

- Lindo Global Solver is designed for the following MINLP problem

$$\min f = f(x_1, x_2, \cdots, x_n)$$

$$s.t$$

$$f_i(x_1, x_2, \cdots, x_n) = 0, \quad i = 1, 2, \cdots, m$$

$$L_j \leq x_j \leq U_j, \quad j = 1, 2, \cdots, n$$

$$x_j \text{ is integer for } j \in J$$

- Constraint types can be any of $\leq$, $=$, or $\geq$
- The objective can be either min or max, or NO objective
- Lindo Global Solver can find a mathematically guaranteed global optimum within predefined tolerance

Lindo Global Solver Overview (Cont'd)

- Lindo Global Solver fully supports all common math functions:
  - Continuous and smooth: $+$, $-$, $\times$, $\ln(x)$, $\log(x)$, e^x, $\sqrt{x}$, etc.
  - Smooth, not quite continuous: x/y, x^y, floor(x), mod(x, y), sign(x), etc.
  - Continuous, not quite smooth: $|x|$, $\max(x, y)$, $\min(x, y)$, etc.
  - Trigonometric: $\sin(x)$, $\cos(x)$, $\tan(x)$, $\arcsin(x)$, $\arccos(x)$, etc.
  - Logical: IF, AND, OR, NOT, EQ, NE, GT, etc.
  - Statistical: psn(x)(Normal CDF), normsinv(x)(inverse of Normal CDF), psl(x)(Normal linear loss function), pel(x, n)(Erlang loss probability), peb(x, n)(Erlang busy probability), pps(x, n)(Poisson CDF), etc.

Lindo Global Solver Overview (Cont'd)

- Lindo Global Solver Methodology:
  - Linearization: functions such as $|x|$, $\max(x, y)$, etc. get linearized using 0/1 variables, solved as MIP
  - Multi-start: try several distinct starting points to quickly find a good local optimum
  - Relaxation and Branch

    1. Relax non-convex/non-smooth problem into linear subproblem; for each arbitrary nonlinear function, given current bounds on variables, automatically construct linear relaxation of the function
    2. Solve the relaxed linear subproblem and pray that solution is feasible to the original model, otherwise branching, i.e. partition the feasible region into two subregions, back to (1)

Lindo Global Solver Overview (Cont'd)

- Core techniques: interval analysis, convex relaxation, algebraic reformulation, constraint propagation, bound tightening, various cut scheme, etc.
- Convexification and Linearization
  - Construct linear envelope to enclose non-convex/non-smooth domain
  - LP is easier to solve
  - Solution provides a good starting point for local MINLP/NLP solver
- Interval Analysis: bounds on the variables and functions
- Constraint Propagation: tighten bounds
- Algebraic Reformulation: tighten bounds and strong relaxations
- IP and NLP preprocessing: reducing model size and complexity

Lindo Global Solver Overview (Cont'd)

- Multi-start local solver finding a "good" upper bound $\bar{f}$ and solution $\bar{x}$
- Model preprocessing, reformulation, bound tightening
- Branch-and-bound iteration
  1. Pop a box from pool. Exit, if pool is empty.
  2. Construct linear relaxed model and solve, obtain lower bound $\hat{f}$ and solution $\hat{x}$
  3. If $\hat{f} \geq \bar{f}$, discard the box and go to (1); If $\hat{x}$ is feasible to original model, update $\bar{f}$ and $\bar{x}$ and go to (1); otherwise continue
  4. If number of cuts not exceeding maximum allowed, add various valid cutting planes to relaxation, and go to (2) for better lower bound
  5. Local solver starting from $\hat{x}$ for updating $\bar{f}$ and $\bar{x}$
  6. Bound tightening procedure
  7. Branching procedure, put resulting boxes into pool.

Identify Convex and Concave Constraints

Based on known convexity and concavity rules to decide convexity of composite function $f \circ g$

- **Rule 1**: If the function f depends on one argument only and is convex on the range of its argument then
  1. If the child function g is linear then $f \circ g$ is convex.
  2. If the child function g is convex and f is monotonic increasing in the interval of g then $f \circ g$ is convex.
  3. If the child function g is concave and f is monotonic decreasing in the interval of g then $f \circ g$ is convex.

Identify Convex and Concave Constraints (Cont'd)

- **Rule 2**: If the function f depends on one argument only and is concave on the range of its argument then

  1. If the child function g is linear then $f \circ g$ is concave.
  2. If the child function g is convex and f is monotonic decreasing in the interval of g then $f \circ g$ is concave.
  3. If the child function g is concave and f is monotonic increasing in the interval of g then $f \circ g$ is concave.

Semidefinite Programming

- Semidefinite Programming allow positive semidefinite matrix to be used in addition to scalar variables.

$$
\begin{aligned}
\min \quad & c^T x + \sum_{j=1}^{p} < \bar{C}_j, \bar{X}_j > \\
s.t. \quad & Ax + \sum_{j=1}^{p} < \bar{A}_j, \bar{X}_j > = b \\
& \bar{X}_j \succeq 0, \quad j = 1, \dots, p
\end{aligned}
$$

Semidefinite Programming (Cont'd)

- Applied to SDP relaxation of quadratic terms,
  - Consider a quadratic term

$$r = x^T Q x, \; -\infty < l \leq x \leq u < +\infty$$

  - Introduce new variables $X_{ij} = x_i x_j$, then

$$r =< Q, X >$$

  - Semidefinite programming (SDP) relaxation:

$$X = xx^T \succeq 0$$

Perspective Reformulation

- Consider model with semi-continuous variables x

$$x \in \{0\} \cup [l, u],$$

or with constraints

$$ly \le x \le uy, \quad y \in \{0, 1\}$$

- The perspective of a function $f(x)$ is

$$\tilde{f}(y, x) = \begin{cases} yf(x/y) & \text{if } y > 0 \\ 0 & \text{if } y = 0 \end{cases}$$

- $\tilde{f}$ is convex if f is convex
- Perspective reformulation provides significantly stronger bounds than continuous relaxation.

Perspective Reformulation (Cont'd)

- Unit commitment problem with convex quadratic cost function

$$
\begin{aligned}
\min \quad & \sum_{i\in I}\sum_{t\in T} h_{it}z_{it} + \sum_{i\in I}\sum_{t\in T}(a_{it}x_{it}^2 + b_{it}x_{it}) \\
s.t. \quad & \sum_{i\in I} x_{it} = d_t \;\; \forall t \in T \\
& l z_{it} \le x_{it} \le u z_{it} \;\; \forall i \in I, \, \forall t \in T \\
& z_{it} \in \{0,1\}, \;\; \forall i \in I, \, \forall t \in T,
\end{aligned}
$$

- Introduce new variables y_{it} and constraints

$$a_{it}x_{it}^2 + b_{it}x_{it} \le y_{it}$$

- Replaced with its perspective

$$a_{it}x_{it}^2 + b_{it}x_{it}z_{it} \le y_{it}z_{it}$$

- Represented using a rotated SOC constraints

$$a_{it}x_{it}^2 \le (y_{it} - b_{it}x_{it})z_{it}$$

Perspective Reformulation (Cont'd)

- Separable quadratic uncapacitated facility location problem

$$\begin{aligned} \min \quad & \sum_{i \in I} c_i z_i + \sum_{i \in I} \sum_{j \in J} q_{ij} x_{ij}^2 \\ s.t. \quad & \mu z_i \leq x_{ij} \leq z_i \;\; \forall i \in I, \forall j \in J \\ & \sum_{i \in I} x_{ij} = 1 \;\; \forall j \in J \\ & z_i \in \{0, 1\}, \forall i \in I. \end{aligned}$$

- Introduce new variables y_{ij} and constraints

$$x_{ij}^2 \leq y_{ij}$$

- Replaced with its perspective

$$x_{ij}^2 / z_i \leq y_{ij}$$

- Represented using a rotated SOC constraints

$$x_{ij}^2 \leq y_{ij} z_{ij}$$

Perspective Reformulation (Cont'd)

- Markowitz Mean-Variance Model with Minimum Buy-in Threshold

$$
\begin{aligned}
\min \quad & x^T Q x \\
s.t. \quad & e^T x = 1 \\
& \alpha^T x \geq \rho \\
& e^T z \leq K \\
& l_i z_i \leq x_i \leq u_i z_i, \ \forall i \in N \\
& z_i \in \{0, 1\}, \ \forall i \in N.
\end{aligned}
$$

- Cannot directly apply perspective reformulation
- Find nonnegative diagonal matrix D, such that

$$R = Q - D \succeq 0$$

- Apply perspective reformulation on term $x^T D x$

Perspective Reformulation (Cont'd)

- Find D using SDP solver

$$\begin{aligned} \max \quad & \sum_{i=1}^{n} d_i \\ s.t. \quad & Q - \sum_{i=1}^{n} d_i (e_i e_i^T) \succeq 0 \\ & d \geq 0, \end{aligned}$$

or the dual

$$\begin{aligned} \min \quad & < Q, X > \\ s.t. \quad & \text{diag}(X) \geq e \\ & X \succeq 0. \end{aligned}$$

Concluding Remarks

- We present a brief overview of LINDO Global Solver
- We present recent developments with LINDO global solver
  - Identify convexity of composite functions based on simple rules.
  - Support semidefinite programming, which is applied to SDP relaxation of quadratic terms, as well as perspective reformulation of convex quadratic terms.
  - Perspective reformulation improves performance on models with semi-continuous variables.