Strong Convex Nonlinear Relaxations of the Pooling Problem

One Relaxation to Rule Them All?

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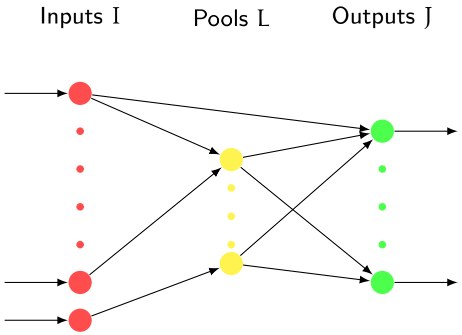
The Pooling Problem



- Nodes $N = I \cup L \cup J$
- Arcs A $(i,j) \in (I \times L) \cup (L \times J) \cup (I \times J)$ on which materials flow
- Material attributes: K
- Arc capacities: u_{ij}
- Node capacities: C_i, $i \in N$
- Attribute requirements β_{kj}, $k \in K, j \in J$

Attribute Blending: Bilinear

- Inputs have associated attribute concentrations $\lambda_{ki},\ k \in K, i \in I$
- The concentration of an attribute in pool is the weighted average of the concentrations of its inputs—This results in bilinear constraints.

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Variables

- x_{ij}: Flow on $(i,j) \in A$
- $q_{i\ell}$: Proportion of flow to pool $\ell \in L$ from input $i \in I$. $(q_{i\ell} = \frac{x_{i\ell}}{\sum_{j \in J} x_{\ell j}})$
  - Note also that $\sum_{i \in I} q_{i\ell} = 1 \quad \forall \ell \in L$
- $w_{ilj} = q_{il} x_{lj}$ (flow from i through pool ℓ to output j)

$$x_{il} = \sum_{j \in J} w_{i\ell j} \quad \forall \ell \in L, j \in J \qquad x_{\ell j} = \sum_{i \in I} w_{i\ell j}$$

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Start Strong!

We use the PQ-formulation (Sahinidis and Tawarmalani (2005) as our starting point

Linearizing Attribute Restriction

- Since $w_{i\ell j} = q_{i\ell} x_{\ell j}$, the attribute constraints have the (linear) form

$$\sum_{i \in I} \lambda_{ki} x_{ij} + \sum_{i \in I} \sum_{\ell \in L} \lambda_{ki} w_{i\ell j} \leq \beta_{kj} \sum_{i \in I \cup L} x_{ij} \quad \forall k \in K, j \in J$$

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- Since $x_{\ell j} = \sum_{i \in I} w_{i\ell j}$, we can write this as

$$\sum_{i \in I} (\lambda_{ki} - \beta_{kj}) x_{ij} + \sum_{i \in I} \sum_{\ell \in L} (\lambda_{ki} - \beta_{kj}) w_{i\ell j} \leq 0 \quad \forall k \in K, j \in J$$

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Upshot

- Improved relaxation will require looking at more of problem!


ONE DOES NOT SIMPLY
CONVEXIFY THE BILINEAR TERMS ALONE
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Our Quest—Seek Simple Sets

- Extract a simple but nontrivial set and attempt to convexify

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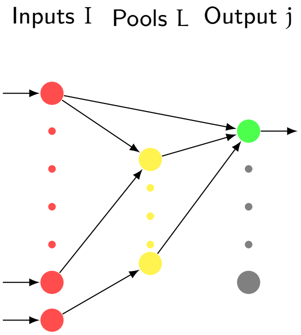
Intuition for Pooling

- Will need to include more than just the nonconvexity $w_{i\ell j} = q_{i\ell}x_{\ell j}$
- Attribute constraints on outputs are important
- Idea: Focus on a single output and attribute

Make It Easier. Focus on Single Output, Single Attribute.



- Fix output j and attribute k.
  - (Drop these indices)
- Relevant constraints

$$\sum_{i \in I} \underbrace{(\lambda_i - \beta)}_{\gamma_i} x_i + \sum_{i \in I} \sum_{\ell \in L} \underbrace{(\lambda_i - \beta)}_{\gamma_i} w_{i\ell} \leq 0$$

$$\sum_{i \in I \cup L} x_i \leq C$$

$$\sum_{i \in I} q_{i\ell} = 1, \qquad \forall \ell \in L$$

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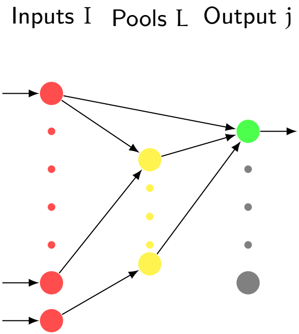

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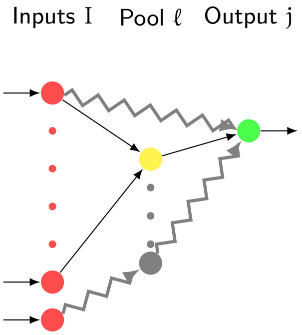
Still Too Hard!

- Try to focus on a single pool $\ell \in L$ and consider rest as a single "by-pass"?

Make it Easier-er. Focus on Single Pool.



- Separate pool $\ell \in L$ from the rest
- Gray squiggles are now aggregated into variables y and z

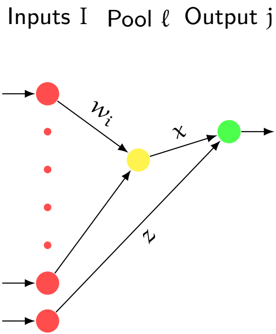
$$\underbrace{\sum_{i \in I} \gamma_i x_i + \sum_{i \in I} \sum_{t \in L \setminus \{\ell\}} \gamma_i w_{it}}_{y} + \sum_{i \in I} \gamma_i w_{i\ell} \leq 0$$

$$\underbrace{\sum_{i \in I \cup L \setminus \{\ell\}} x_i}_{z} + x_\ell \leq C,$$

- $z \geq 0, \quad \underline{\gamma} z \leq y \leq \overline{\gamma} z,$

$$\overline{\gamma} = \max_{i \in I} \gamma_i \geq 0 \underline{\gamma} = \min_{i \in I} \gamma_i \leq 0$$

Single Output, Single Pool Set



- Now also drop the index ℓ for the pool:

$$
\begin{aligned}
y + \sum_{i \in I} \gamma_i w_i &\leq 0, \\
z + x &\leq C, \\
\sum_{i \in I} q_i &= 1, \\
w_i &= q_i x, \quad \forall i \in I \\
\underline{\gamma} z \leq y &\leq \overline{\gamma} z \\
x, w, z, q &\geq 0
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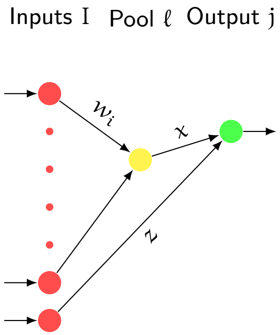

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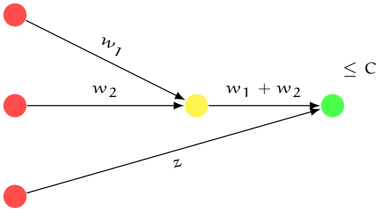
x, w, z, q &\geq 0
\end{aligned}
$$

Argh! Still too hard

- Let's assume only two inputs into the pool

Easiest—Single Output, Single Pool, Two Input Set

$$
\begin{aligned}
y + \gamma_1 w_1 + \gamma_2 w_2 &\le 0, \\
z + w_1 + w_2 &\le C, \\
q_1 + q_2 &= 1, \\
w_1 &= q_1(w_1 + w_2), \\
w_2 &= q_2(w_1 + w_2), \\
\underline{\gamma} z \le y &\le \overline{\gamma} z \\
w_1, w_2, q_1, q_2, z &\ge 0
\end{aligned}
$$



- We assume $\gamma_1 < 0$, $\gamma_2 > 0$—Other cases are simpler
- Let P be set of points satisfying above

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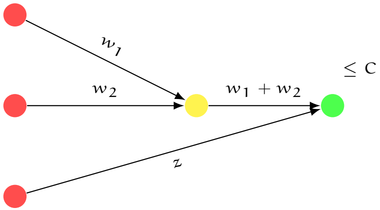

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One Relaxation To Rule Them All

- We would like to understand $\mathrm{conv}(P)$

The Fellowship of the Pool

- Our “magic” for finding inequalities for nonlinear sets

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Teamwork!

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3. Jim: Identify parameterized set of inequalities in PORTA output, and prove validity of conjectured inequality
4. Jeff: Confuse Things. Attempt Jokes for Talks.

Extreme Points $1, \ldots, 8$

w_1	w_2	q_1	q_2	z	y
0	0	1	0	0	0
0	0	0	1	0	0
0	0	1	0	C	0
0	0	0	1	C	0
0	0	1	0	C	$\underline{\gamma}C$
0	0	0	1	C	$\underline{\gamma}C$
C	0	1	0	0	0
$\frac{\gamma_2 C}{\gamma_2 - \gamma_1}$	$\frac{-\gamma_1 C}{\gamma_2 - \gamma_1}$	$\frac{\gamma_2}{\gamma_2 - \gamma_1}$	$\frac{-\gamma_1}{\gamma_2 - \gamma_1}$	0	0

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0	0	1	0	C	$\underline{\gamma}C$
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$\frac{\gamma_2 C}{\gamma_2-\gamma_1}$	$\frac{-\gamma_1 C}{\gamma_2-\gamma_1}$	$\frac{\gamma_2}{\gamma_2-\gamma_1}$	$\frac{-\gamma_1}{\gamma_2-\gamma_1}$	0	0

- Let $q_1 = \alpha$, $\kappa_\alpha = \alpha\gamma_1 + (1-\alpha)\gamma_2$

Extreme Points $9,\ldots,\infty$, $q_1 = \alpha$

	w_1	w_2	q_1	q_2	z	y
$\alpha \in [0, \gamma_2/(\gamma_2-\gamma_1)] (\kappa_\alpha \geq 0)$	$\frac{-\alpha\underline{\gamma}C}{\kappa_\alpha-\underline{\gamma}}$	$\frac{-(1-\alpha)\underline{\gamma}C}{\kappa_\alpha-\underline{\gamma}}$	α	$1-\alpha$	$\frac{\kappa_\alpha C}{\kappa_\alpha-\underline{\gamma}}$	$\frac{\underline{\gamma}\kappa_\alpha C}{\kappa_\alpha-\underline{\gamma}}$
$\alpha \in [\gamma_2/(\gamma_2-\gamma_1), 1] (\kappa_\alpha < 0)$	$\frac{\alpha\overline{\gamma}C}{\overline{\gamma}-\kappa_\alpha}$	$\frac{(1-\alpha)\overline{\gamma}C}{\overline{\gamma}-\kappa_\alpha}$	α	$1-\alpha$	$\frac{-\kappa_\alpha C}{\overline{\gamma}-\kappa_\alpha}$	$\frac{-\overline{\gamma}\kappa_\alpha C}{\overline{\gamma}-\kappa_\alpha}$

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- Consider polyhedron defined by finite subset of these extreme points
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- Stare at them a long time...

```
 INEQUALITIES_SECTION
( 1) - 50x1- 5x2- 85x3- 12x4- x5+ x6 <= -150
( 2) - 50x1- x2- 87x3- 12x4- x5+ x6 <= -148
( 3) - 50x1- 5x2- 85x3- 2x4- 7x5+ x6 <= -146
( 4) - 40x1-10x2- 85x3- 12x4- x5+ x6 <= -145
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( 6) - 40x1-10x2- 85x3- 2x4- 7x5+ x6 <= -141
( 7) - 50x1- 5x2- 13x3- 60x4- x5+ x6 <= -126
( 8) - 50x1- x2- 15x3- 60x4- x5+ x6 <= -124
( 9) - 40x1-10x2- 13x3- 60x4- x5+ x6 <= -121
( 10) - 8x1-10x2- 93x3- 12x4- x5+ x6 <= -121
( 11) - 50x1- 5x2- 3x3- 60x4- 5x5+ x6 <= -120
( 12) - 10x1- x2- 97x3- 12x4- x5+ x6 <= -118
( 13) - 50x1- x2- 5x3- 60x4- 5x5+ x6 <= -118
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( 15) - 8x1- 2x2- 97x3- 12x4- x5+ x6 <= -117
( 16) - 2x1-10x2- 93x3- 13x4- x5+ x6 <= -116
( 17) - 40x1-10x2- 3x3- 60x4- 5x5+ x6 <= -115
( 18) - 10x1- x2- 97x3- 2x4- 7x5+ x6 <= -114
( 19) - 8x1- 2x2- 97x3- 2x4- 7x5+ x6 <= -113
( 20) - 4x1- x2- 97x3- 13x4- x5+ x6 <= -113
( 21) - 2x1- 2x2- 97x3- 13x4- x5+ x6 <= -112
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"If you sit on the door-step long enough, I daresay you will think of something."

Bilbo

Valid Inequalities - Type I

Theorem

The following inequaliy is valid for P

$$\left((\gamma_1 - \underline{\gamma})w_1 + (\gamma_2 - \underline{\gamma})w_2\right)^2 \leq -\underline{\gamma}C((\gamma_1 - \underline{\gamma})q_1 + (\gamma_2 - \underline{\gamma})q_2)(w_1 + w_2).$$

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- The inequality is supported by the "small alpha" extreme points
  - By-pass is necessary to make up for excess concentration from the pool
- The inequality is second-order cone representable

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Easy Proof

- Aggregate linear inequalities $\Rightarrow (\gamma_1 - \underline{\gamma})w_1 + (\gamma_2 - \underline{\gamma})w_2 \leq -\underline{\gamma}C$
- Observe:
  $(\gamma_1 - \underline{\gamma})w_1 + (\gamma_2 - \underline{\gamma})w_2 = (\gamma_1 - \underline{\gamma})q_1(w_1 + w_2) + (\gamma_2 - \underline{\gamma})q_2(w_1 + w_2)$
- Multiply LHS with LHS, RHS with RHS

Victory Is Ours—It Works for Multiple Inputs

- The "same" proof demonstrates how to create a valid inequality for multiple inputs

Theorem

The following inequality is valid for P with multiple inputs I:

$$\left(\sum_{i \in I} (\gamma_i - \underline{\gamma}) w_i\right)^2 \leq -\underline{\gamma} C \left(\sum_{i \in I} (\gamma_i - \underline{\gamma}) q_i\right)\left(\sum_{i \in I} w_i\right).$$

Valid Inequalities—Type II

Theorem

The following inequality is valid for P if $y > 0$:

$$\bar{\gamma} w_1 + y + \frac{\gamma_1 w_2 y}{(\gamma_2 - \gamma_1) w_2 + y} \leq \bar{\gamma} C q_1$$

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- Second-order cone representable for the case $y > 0$

- The inequality is only valid if $y \leq 0$. Let

$$f(y, w_2) := \begin{cases} 0 & y \leq 0 \\ y + \frac{\gamma_1 w_2 y}{(\gamma_2 - \gamma_1) w_2 + y} & y > 0. \end{cases}$$

- $f(y, w_2)$ is convex, and the following inequality is valid for P:

$$f(y, w_2) \leq \overline{\gamma}(C q_1 - w_1)$$

- Reduces to inequality from last slide when $y > 0$, to $w_1 \leq C q_1$ when $y \leq 0$

A Conjecture

"I don't know, and I would rather not guess."

Frodo

P

$$
\begin{aligned}
w_1 &= q_1(w_1 + w_2)\\
w_2 &= q_2(w_1 + w_2)\\
y + \gamma_1 w_1 + \gamma_2 w_2 &\le 0\\
z + w_1 + w_2 &\le C\\
q_1 + q_2 &= 1\\
\underline{\gamma} z \le y &\le \overline{\gamma} z\\
w_1, w_2, q_1, q_2, z &\ge 0
\end{aligned}
$$

conv(P)

$$
\begin{aligned}
&\Big(\sum_{i=1}^{2} (\gamma_i - \underline{\gamma}) w_i\Big)^2 \le\\
&\qquad -\underline{\gamma} C \Big(\sum_{i=1}^{2} (\gamma_i - \underline{\gamma}) q_i\Big)\Big(\sum_{i=1}^{2} w_i\Big)
\end{aligned}
$$

$$
\begin{aligned}
f(y, w_2) &\le \overline{\gamma}(C q_1 - w_1)\\
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Computational Results

"No One Believes Me"

J.R.R. Tolkein

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- McCormick relaxation of PQ-Formulation (LP) versus LP+Type 1 inequalities, Optimal Value
- Instances from the literature (exclude those for which LP yields optimal value)
- These are small instances

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TA-DA!

Instance	z^{PQ}	$z^{\text{Type I}}$	z^*
adhya1	-770.0	-738.7	-549.8
adhya2	-572.3	-569.9	-549.8
bental4	-550.0	-525.0	-450.0
haverly1	-500.0	-475.0	-400.0
haverly2	-1000.0	-866.7	-600.0
rt2	-6034.9	-6034.9	-4391.8

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- These results are honestly not that great, but there is more to be done!

"Despair is only for those who see the end beyond all doubt. We do not."

Gandalf

Conclusions

"All's well that ends better"

The Gaffer

- Even with one output, one pool, two inputs
- Convex hull is not polyhedral

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Continuing Quest

- Prove Convex Hull result
- Extend Type-II inequality to multiple inputs
- Better approximation of by-pass (e.g., upper bounds on different inputs)
- Multiple pools?