Valid Inequalities for Potential-Constrained Network Design

Hyemin Jeon
Jeff Linderoth
Jim Luedtke

Dept. of ISyE
UW-Madison
jrluedt1@wisc.edu

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Motivating Application: Power Flow

- Power Network: $(\mathcal{N}, \mathcal{A})$ with
  - $\mathcal{G} \subset \mathcal{N}$: generation nodes
  - $\mathcal{D} \subset \mathcal{N}$: demand nodes

Motivating Application: Power Flow

- Power Network: (N, A) with
  - $G \subset N$: generation nodes
  - $D \subset N$: demand nodes

Variables

- p_i: (Real) power inject at generator $i \in G$
- x_{ij}: (Real) power flow on line $(i, j) \in A$
- θ_i: Voltage angle at node $i \in N$

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DC Power Flow Assumption

- The (real) power transmit over line $(i, j) \in A$ is proportional to angle differences at the endpoint nodes $i \in N$ and $j \in N$.

$$x_{ij} = \alpha_{ij}(\theta_j - \theta_i)$$

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More general potential constraints on flow (e.g., water and gas distribution):

$$g_{ij}(x_{ij}) = \theta_j - \theta_i$$

(DC) Optimal Power Flow

$$\min_{x,p,\theta} \sum_{i \in G} c_i p_i$$

$$\text{s.t.} \quad \sum_{j:(i,j)\in E} x_{ij} - \sum_{j:(j,i)\in E} x_{ji} = \begin{cases} p_i & \forall i \in G \\ d_i & \forall i \in D \\ 0 & \forall i \in N \setminus G \setminus D \end{cases}$$

$$-U_{ij} \le x_{ij} \le U_{ij} \quad \forall (i,j) \in E$$

$$\underline{p}_i \le p_i \le \overline{p}_i \quad \forall i \in G$$

$$x_{ij} = \alpha_{ij}(\theta_j - \theta_i) \quad \forall (i,j) \in E$$

$$p_i \in \mathbb{R}_+ \quad \forall i \in G$$

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- x, θ need not be ≥ 0
- Bounds on x, but no a priori bounds on θ (Usually derived from bounds on x)

Transmission Switching

Tradeoff

- Having Edges/Lines Allows You to Send Flow:

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- Fisher, O'Neill & Ferris ('08) show that efficiency improved by optimally switching off transmission lines

Max Lines Off	% Improvement
1	6.3%
2	12.4%
3	19.9%

- Same problem structure appears in transmission network design problems and in (nonlinear) gas/water network design problems

Switching Off Lines

- Regular Flow Constraints

$$
\begin{aligned}
x_{ij} &= \alpha_{ij}(\theta_j - \theta_i) && \forall (i,j) \in E \\
-U_{ij} \le x_{ij} &\le U_{ij} && \forall (i,j) \in E
\end{aligned}
$$

Switching Off Lines

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$$x_{ij} = \alpha_{ij}(\theta_j - \theta_i) \quad \forall (i,j) \in E$$
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- Switched Flow Constraints

$$x_{ij} = \alpha_{ij} z_{ij} (\theta_j - \theta_i) \quad \forall (i,j) \in E$$

MILP Formulation

- If θ_i have bounds then one can write an MILP formulation (Fisher, O'Neil, and Ferris '08).
- $z_{ij} = 1 \Leftrightarrow$ line $(i,j) \in A$ is used

$$\min_{x,p,\theta,z} \quad \sum_{i \in G} c_i p_i$$

$$\text{s.t.} \quad \sum_{j:(i,j)\in E} x_{ij} - \sum_{j:(j,i)\in E} x_{ij} = \begin{cases} p_i & \forall i \in G \\ d_i & \forall i \in D \\ 0 & \forall i \in N \setminus G \setminus D \end{cases}$$

$$-U_{ij} z_{ij} \le x_{ij} \le U_{ij} z_{ij} \quad \forall (i,j) \in E$$

$$\alpha_{ij}(\theta_i - \theta_j) - x_{ij} + M(1 - z_{ij}) \ge 0 \quad \forall (i,j) \in E$$

$$\alpha_{ij}(\theta_i - \theta_j) - x_{ij} - M(1 - z_{ij}) \le 0 \quad \forall (i,j) \in E$$

$$-L_i \le \theta_i \le L_i \quad \forall i \in N$$

$$z_{ij} \in \{0,1\} \quad \forall (i,j) \in E$$

This is a Hard Problem

- Hedman, Ferris, O'Neill, Fisher, Oren, (2010) state

  > *"When solving the transmission switching problem, ... the techniques for closing the optimality gap, specifically improving the lower bound, are largely ineffective."*

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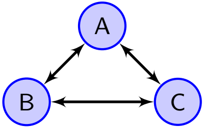
Insight #1

- The problem looks like an (integer) multicommodity flow problem
- With the additional "line-voltage" constraints:

$$x_{ij} = \alpha_{ij}(\theta_j - \theta_i)$$

Key (Simple) Insight?!



- Assume (WLOG) that $\alpha_{ij} = 1$
  - We can just set $x_{ij} = \alpha_{ij} x'_{ij}$ and scale u_{ij} by α_{ij}
- Then we have...

$$\begin{aligned} x_{AB} &= \theta_B - \theta_A \\ x_{BC} &= \theta_C - \theta_B \\ x_{CA} &= \theta_A - \theta_C \\ \hline x_{AB} + x_{BC} + x_{CA} &= 0 \end{aligned}$$

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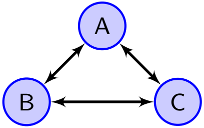

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IP People Like Simple Sets

- Directed cycle $G = (V, C)$, with $V = [n]$, $C = \{(i, i+1) : \forall i \in [n-1]\} \cup \{(n, 1)\}$:

$$\mathcal{C} = \Big\{ (x, \theta, z) \in \mathbb{R}^{2n} \times \{0,1\}^n : -u_{ij} \le x_{ij} \le u_{ij} \ \forall (i,j) \in C$$
$$z_{ij}(\theta_i - \theta_j) = x_{ij} \ \ \forall (i,j) \in C \Big\}$$

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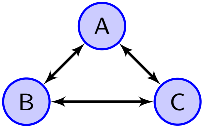

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- Even though $\mathcal{C}$ has the "nonlinear" equations $z_{ij}(\theta_i - \theta_j) = x_{ij}$, it is the union of 2^n polyhedra, so $\operatorname{cl}\operatorname{conv}(\mathcal{C})$ is a polyhedron.

First Result

Theorem

For $S \subseteq C$ such that $u(S) > u(C \setminus S)$, the cycle inequalities

$$x(S) + \sum_{a \in C} \beta_a^S z_a \leq b^S \quad (1)$$

$$-x(S) + \sum_{a \in C} \beta_a^S z_a \leq b^S \quad (2)$$

are valid for $\mathcal{C}$, where

$$\beta_a^S = u(S \setminus a) - u(C \setminus S) \quad \forall a \in C$$

$$b^S = (n-1)(2u(S) - u(C))$$

- Similar result has been obtained by Santanu Dey, Burak Kocuk, and Andy Sun (Georgia Tech).

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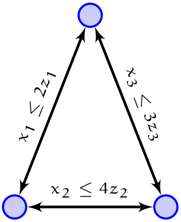
Cycle Inequalities, Example



$$x_1 + x_2 + z_1 - z_2 + 3z_3 \leq 6 \quad S = \{1, 2\}$$
$$x_1 + x_3 - z_1 + z_2 - 2z_3 \leq 2 \quad S = \{1, 3\}$$
$$x_2 + x_3 + 5z_1 + z_2 + 2z_3 \leq 10 \quad S = \{2, 3\}$$
$$x_1 + x_2 + x_3 + 7z_1 + 5z_2 + 6z_3 \leq 18 \quad S = \{1, 2, 3\}$$

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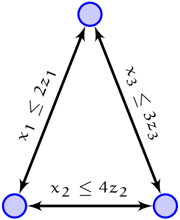

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Logic Enforced

- For $S = \{1, 2\}$, if $z_1 = z_2 = 1$, then

$$x_1 + x_2 \leq \begin{cases} 6 & z_3 = 0 \\ 3 & z_3 = 1 \end{cases}$$

- For $S = \{1, 3\}$, if $z_1 = z_3 = 1$, then

$$x_1 + x_3 \leq \begin{cases} 5 & z_2 = 0 \\ 4 & z_2 = 1 \end{cases}$$

Strength of cycle inequalities

Theorem

If $S \subseteq C$, and $u(C \setminus S) < u(S)$, then the cycle inequalities (CI) are facet-defining for $\mathrm{cl\,conv}(\mathcal{C})$.

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- Something we've conjectured, and Dey, Kocuk, and Sun have proved:

$$\mathrm{cl\,conv}(\mathcal{C}) = \Big\{(x, \theta, z) \in \mathbb{R}^{3n} : -u_{ij}z_{ij} \le x_{ij} \le u_{ij}z_{ij} \quad \forall (i,j) \in C$$
$$z_{ij} \le 1 \quad \forall (i,j) \in C$$
$$x(S) + \sum_{a \in C} \beta_a^S z_a \le b^S \quad \forall S \subseteq C : u(S) > u(C \setminus S)$$
$$-x(S) + \sum_{a \in C} \beta_a^S z_a \le b^S \quad \forall S \subseteq C : u(S) > u(C \setminus S)\Big\}$$

Separation

- Given solution $\hat{x} \in \mathbb{R}^n_+, \hat{z} \in [0,1]^n$, the separation problem for (CI) is

$$\max_{C \subseteq A : C \text{ is a cycle}} \max_{S \subseteq C : 2u(S) \geq u(C)} \{\hat{x}(S) + (\beta^S)^\top \hat{z} - b^S\}.$$

- Observation: If $\sum_{a \in C} \hat{z}_a \leq |C| - 1$, then $(\hat{x}, \hat{z})$ cannot be violated by any (CI)

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- Do (1) by (truncated) enumeration
- Given C, algebra shows that (2) is equivalent to a knapsack problem:
  - $\hat{\lambda} = |C| - 1 - \sum_{a \in C} \hat{z}_a$
  - $\hat{v}_a = \hat{x}_a + u_a \hat{z}_a - 2u_a (\sum_{e \in C \setminus a} (1 - \hat{z}_e))$

$$v = \max_{y \in \{0,1\}^n} \left\{ \sum_{a \in C} \hat{v}_a y_a : \sum_{a \in C} u_a y_a \geq \frac{1}{2} u(C) \right\}$$

  - If $v + u(C)\hat{\lambda} > 0$, then (CI) is violated by $(\hat{x}, \hat{z})$

Test Problem

- Power grid network design problem.
- One (expensive) generator can supply power to n nodes
- Possibility to "plug in" up to $n/5$ cheaper generators, with fixed cost of constructing new lines
- Also can do transmission switching

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- Ten instances (each) of size $n = 30$, $n = 50$.
- Run CPLEX for one hour, record, initial LP Gap, Final LP Gap, and Final Gap
- Report (arithmetic) averages
- All Gaps taken w.r.t. best feasible solution found

Computational Results

CPLEX Cuts Turned On—Gap %

		No (CI)		With (CI)	
n	LP	Root	Final	Root	Final
30	10.46	9.52	9.16	9.09	8.90
50	11.88	11.46	11.37	11.14	11.10

	No (CI)	With (CI)	
n	#node	#node	# cuts
30	67928.2	1525.5	2074.8
50	6202.3	223.0	759.6

Conclusions

- IP-based approach for improving lower bounds in DC-approximated power network design (e.g., transmission switching)
- (CI) extend to any application where "potential" is preserved around a cycle, and where potential difference and flow are related by a (possibly nonlinear) equation

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Up Next

- Collaborate with Dey, Kocuk, and Sun
- Improve separation routine, test on more problem classes
- Consider using a multi-commodity formulation for improved flow upper bounds
- Study more complicated structures besides cycles?
- Extend to potential preserved, but nonlinear relationship between potential and flow—Gas and Water Network design