A Polyhedral Frobenius Theorem with Applications to Integer Optimization in Variable Dimension

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  - d fixed, f convex (Grötschel, Lovasz, Schrijver '1988)
  - d fixed, f and constraints quasi-convex polynomials (Khachiyan, Porkolab '2000)
  - $d = 2$ and f polynomial of degree two (Del Pia, Weismantel '2014)

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  - Separable-convex integer programming (Hochbaum, Shanthikumar '1990)
  - N-fold integer programming (De Loera, Hemmecke, Onn, Weismantel '2008)
  - Dynamic programming

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Question: under which conditions on the input is this problem tractable?

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Theorem. There is an algorithm that solves the non-linear optimization problem

$$\min \{ f(Wx) \, : \, Ax \leq b, \; x \in \mathbb{Z}^n \}.$$

The number of oracle calls it performs (to the optimization and fiber oracles) is polynomial in n, ω and Δ.

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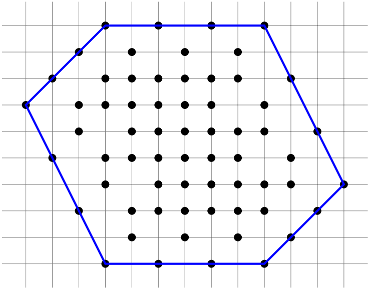
$[0,3]^3 \cap \mathbb{Z}^3$

$$W = \begin{pmatrix} 1 & 2 & 1 \\ -2 & 0 & 1 \end{pmatrix}$$



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- W has HNF Identity, and
- $C(W) = \{W\lambda \ : \ \lambda \geq 0\}$ is a pointed cone.

Let $v = W\mathbb{1}$. The *diagonal Frobenius number* $F(W)$ is defined as the smallest integer t such that

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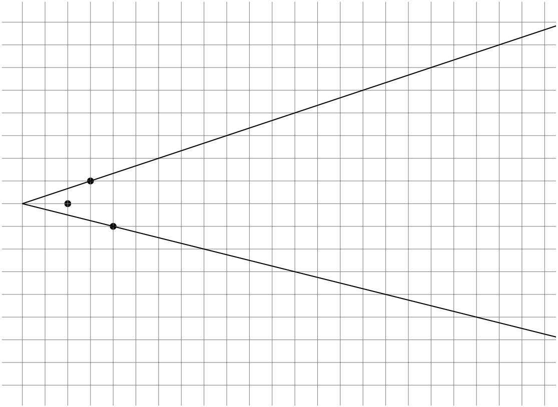

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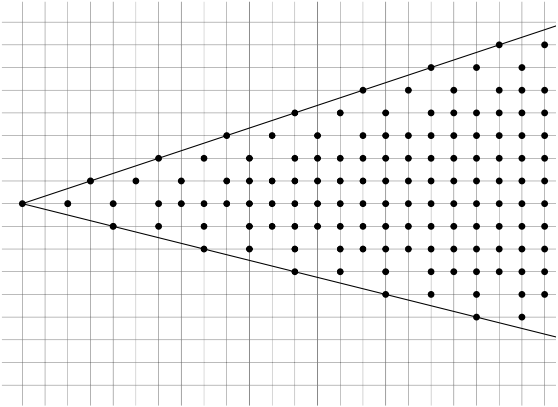

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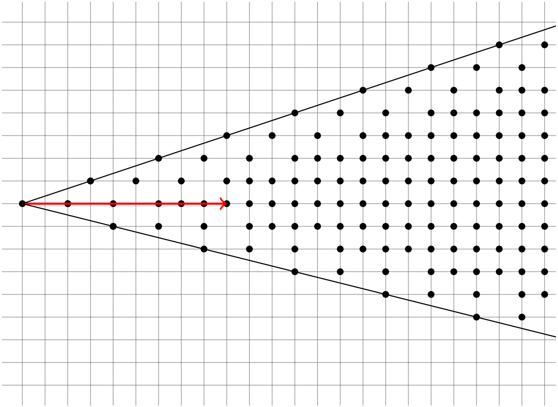

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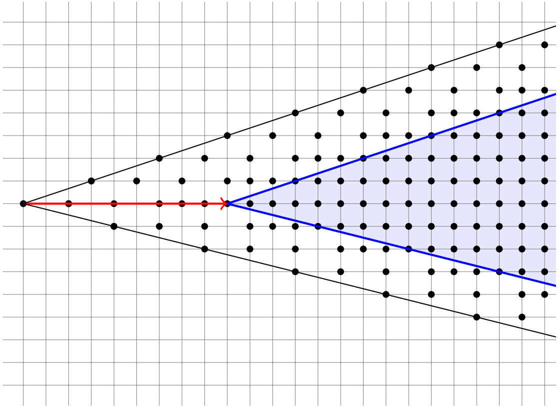

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Theorem (Aliev, Henk 2010).

$$F(W) \leq \frac{(m-d)\sqrt{m}}{2}\sqrt{\det(WW^T)}.$$

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- For fixed d, the bound is polynomial in the unary encoding of W.

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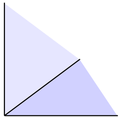
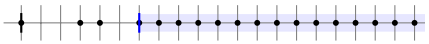
$P = \mathbb{R}^n_+$ 

$w \in \mathbb{Z}^n$ 

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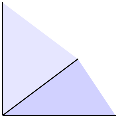

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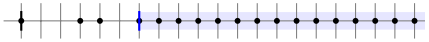

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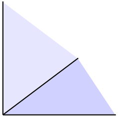

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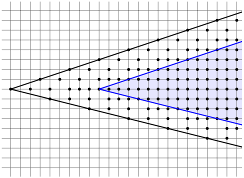

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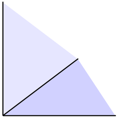

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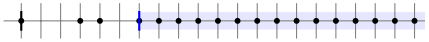

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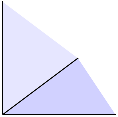

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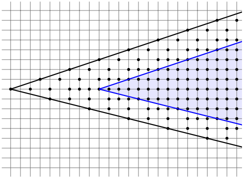

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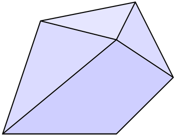

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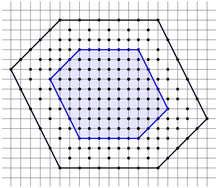

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Definition (δ-regular set). We call a set $\mathcal{S} \subset \mathbb{Z}^d$ δ-regular, with respect to a region $B \subset \mathbb{R}^d$, if there exists a family of full-dimensional affine sub-lattices $\Lambda_1, \cdots, \Lambda_k$ of $\mathbb{Z}^d$ with determinants $\det(\Lambda_i) \leq \delta$ such that

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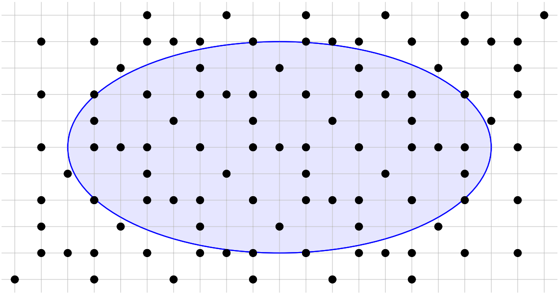

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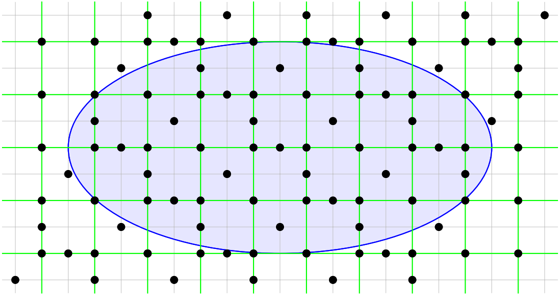

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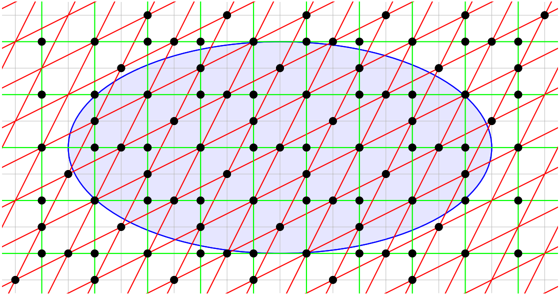

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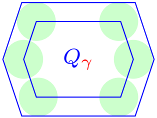

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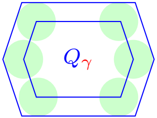

Theorem. The set $\mathcal{R}$ is δ-regular with respect to the polyhedron Q_γ, where γ and δ are bounded polynomially in Δ, $\|W\|_{max}$ and n.

The Algorithm

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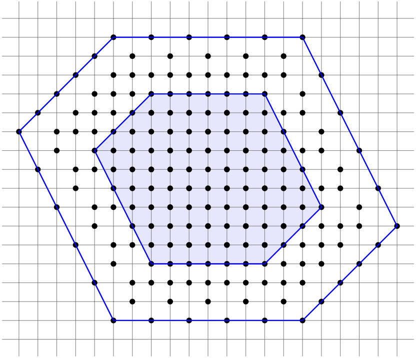

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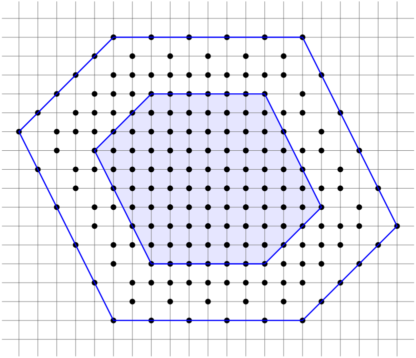

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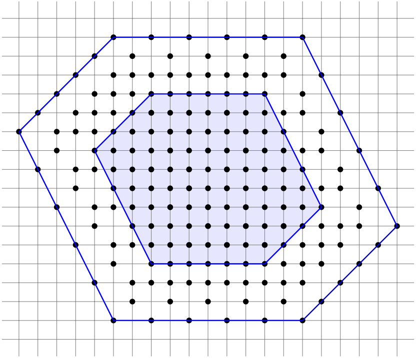

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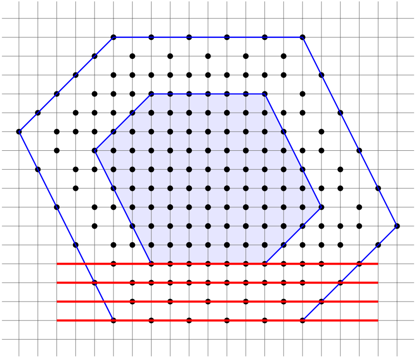

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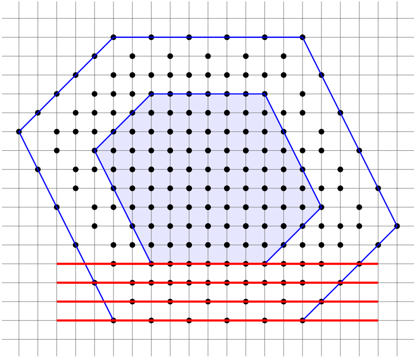

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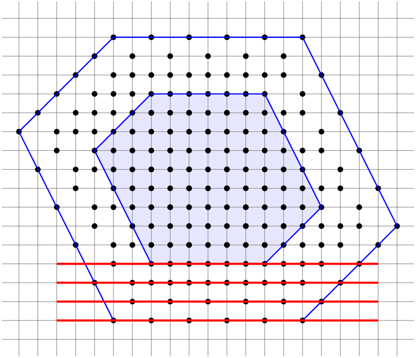

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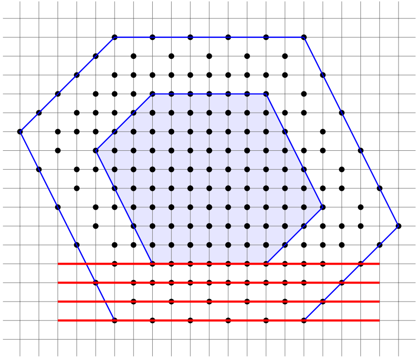

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Thank You!