Centre for Process Systems Engineering

Imperial College London

Solution strategies for mp-MILP and mp-MIQP problems

Richard Oberdieck
Efstratios N. Pistikopoulos

Acknowledgment

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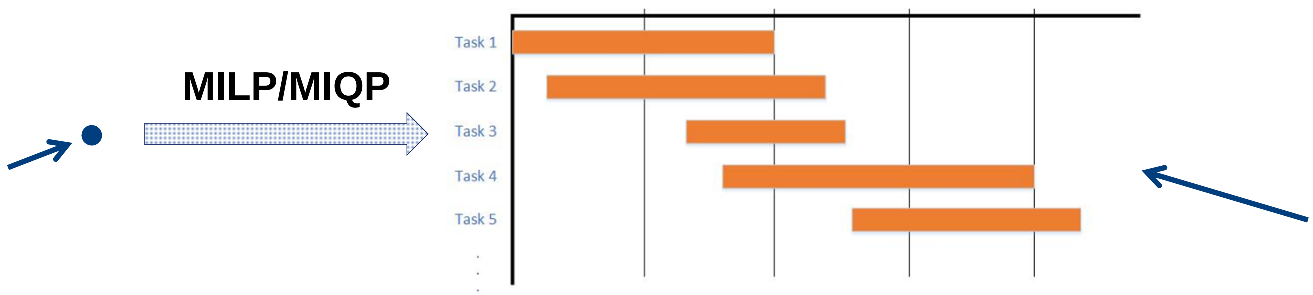
Areas of Interest – Scheduling under Uncertainty

Scheduling

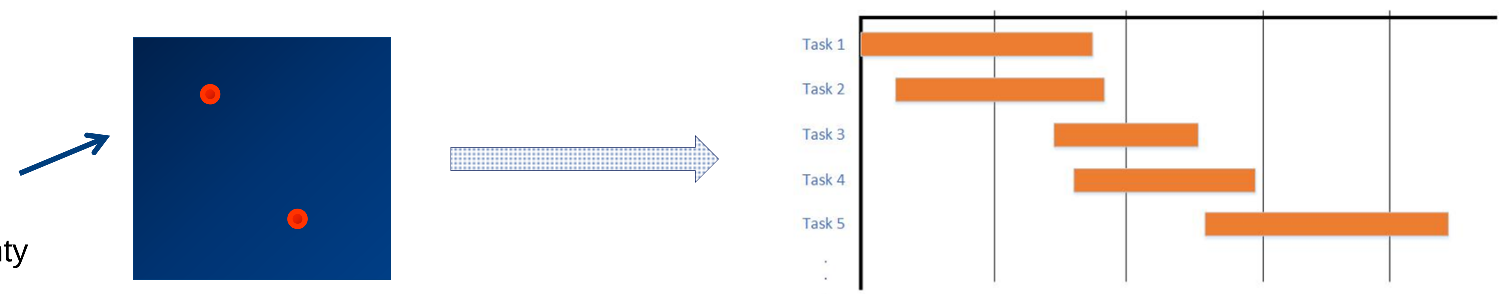
Nominal value of uncertainty



Scheduling policy

Scheduling under Uncertainty

Bounded Uncertainty Space



Scheduling policy changes!

Areas of Interest – Scheduling under Uncertainty

How can we consider the presence of uncertainty preventively?

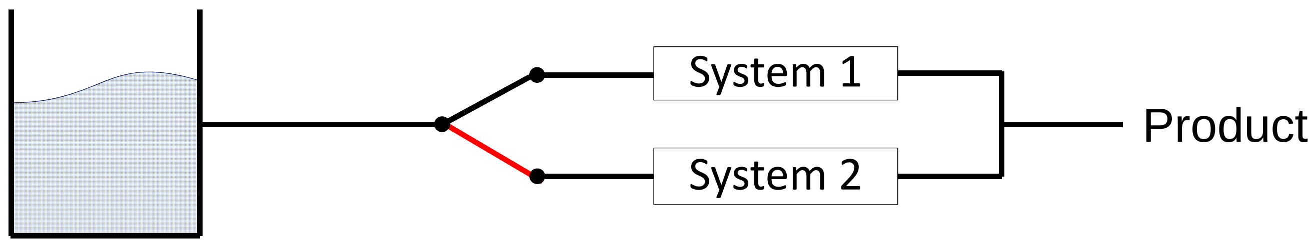
Possible "preventive" scheduling approaches

- **Stochastic Programming:** Use stochastic information to solve for the expected value (only if available)
- **Robust optimization:** Robustify against worst-case uncertainty realization
- **Multiparametric programming:** Solve the optimization problem for the whole state and parameter space

Areas of Interest – Hybrid Control

Hybrid Control: **Control a system consisting of continuous and discrete variables**

Example



Online Hybrid Control  **MILP / MIQP**

Computational Effort!

Areas of Interest – General Problem Formulation

- **All of these problems eventually result in a MILP or MIQP of the general form**

$$
\begin{aligned}
z(\theta) = {} & \min_{\omega} \left(Q^T\omega + c\right)^T \omega \\
\text{s.t.} \quad & A\omega \leq b \\
& \omega \in \Omega = \mathbb{R}^n \times \{0,1\}^m
\end{aligned}
$$

- **If Q does not contain diagonal elements, then it is a MILP. If it does, then it is a MIQP.**

mp-MIQP problems – Problem Formulation

- **When uncertainty is considered in these problems, we obtain mp-MILP or mp-MIQP problems of the general form**

$$
\begin{aligned}
z(\theta) = \min_{\omega} \ & \left(Q^T \omega + P^T \theta + c \right)^T \omega \\
\text{s.t.} \ & \left(\sum_{i=1}^{q} \theta_i \left(A_i + E_i \right) \right) \omega \leq b + F\theta \\
& \omega \in \Omega = \mathbb{R}^n \times \{0,1\}^m \\
& \theta \in \Theta = \left\{ \theta \in \mathbb{R}^q \,\middle|\, \theta_l^{min} \leq \theta_l \leq \theta_l^{max}, l = 1, \ldots, q \right\}
\end{aligned}
$$

- **The main challenges are**
  - How to treat the (possible) non-convexities
  - How to handle the binary/integer variables

mp-MIQP problems – Solution characterization

- **Consider the general mp-QP**

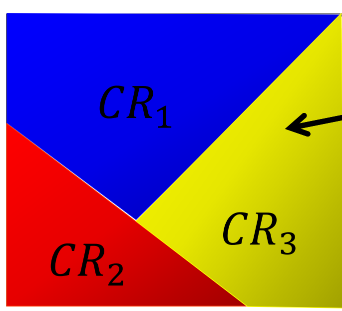
$$
\begin{aligned}
z(\theta) = \min_{x}\ & (Q_x x + H\theta + c)^T x \\
\text{s.t.}\quad & Ax \le b + F\theta \\
& x \in \mathbb{R}^n \\
& \theta \in \Theta = \{\theta \in \mathbb{R}^q \,|\, \theta_l^{min} \le \theta_l \le \theta_l^{max}, l = 1, \dots, q\}
\end{aligned}
$$

- **Note: The term $\boldsymbol{\theta^T H^T x}$ can be avoided using the z-Transformation $\boldsymbol{z = x + Q^{-1} H^T \theta}$.**
- **If $\boldsymbol{Q > 0}$, then the solution to this problem is given by**

$$\mathrm{x}(\theta) = K_i\theta + r_i \quad \text{if } \theta \in CR_i$$

with

$$CR_i = CR_i^A\theta + CR_i^b$$

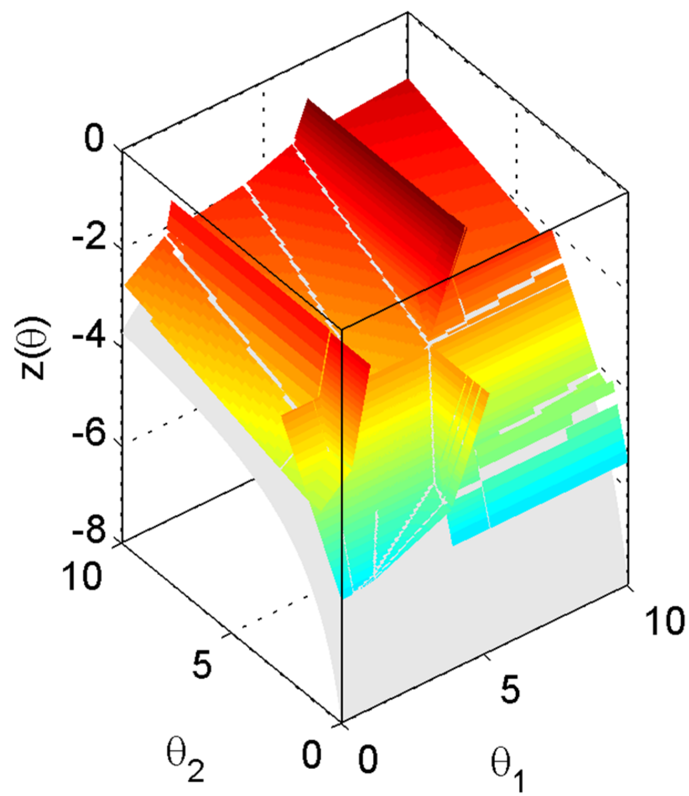


Critical Region 3 with the solution
$\mathrm{x}(\theta) = K_3\theta + r_3$

mp-MILP/mp-MIQP problems – Nonconvexity

- **However, when integer variables or uncertainty in the constraint matrix is present, then the critical regions might be described by non-affine inequalities!**

⇒ **Nonconvexity**

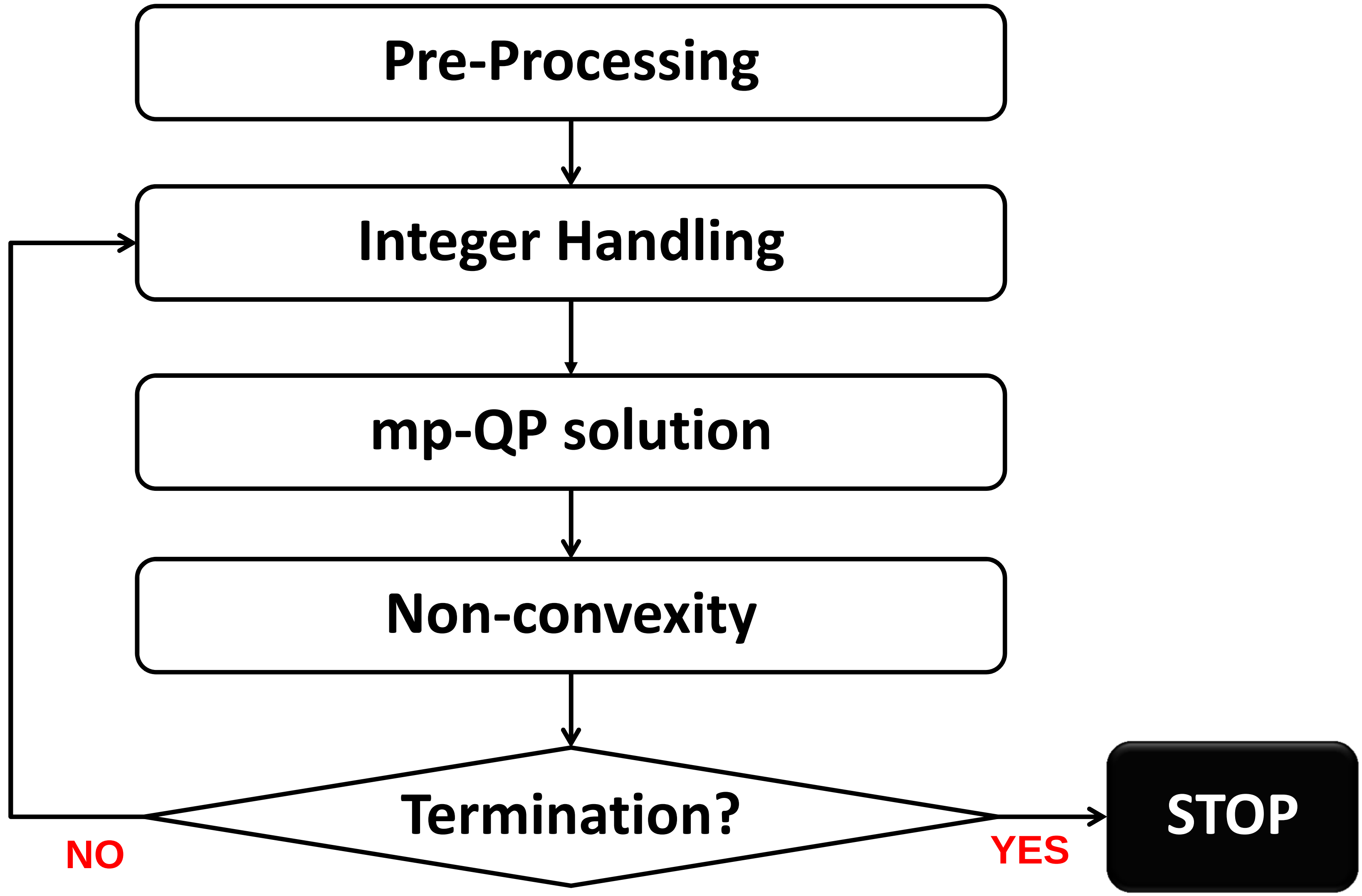


Multiparametric Programming – An Overview (Pistikopoulos, 2012)

Topic	Contribution
mp-LP	Gal and Nedoma (1972), Gal (1975), Acevedo (1996), Dua, Bozinis and Pistikopoulos (2002)
mp-QP	Bemporad et al. (2002), Dua et al. (2002), Tøndel et al. (2003), Spjøtvold et al. (2006), Gupta et al. (2011), Feller and Johanson (2013)
mp-NLP	Fiacco (1976), Bank et al. (1983), Acevedo (1996), Dua and Pistikopoulos (1998)
mp-MILP	Acevedo and Pistikopoulos (1997), Dua and Pistikopoulos (2000), Li and Ierapetritou (2007), Wittmann-Hohlbein and Pistikopoulos (2012), Oberdieck et al. (2014), Mitsos and Barton (2009)
mp-MIQP	Dua et al. (2002), Oberdieck et al. (2014), Axehill et al. (2014)
mp-MINLP	Dua and Pistikopoulos (1999), Dua et al. (2004), Dominguez and Pistikopoulos (2013)

mp-MPC	Bemporad et al. (2002), Sakizlis et al. (2003), Kouramas et al. (2011)
mp-Scheduling	Wittmann-Hohlbein and Pistikopoulos (2013), Kopanos and Pistikopoulos (2014)
Robust mp-MPC	Bemporad et al. (2003), Sakizlis et al. (2004), Faisca et al. (2008)

mp-MIQP problems – Solution framework



mp-MIQP Framework – Pre-Processing

Pre-Processing

1) Initialization of Algorithm

Set the constants and options needed for the execution of the algorithm

2) Robustification of mp-MIQP problem

Select certain parameters in the problem formulation and robustify the problem according to Wittmann-Hohlbein and Pistikopoulos (2013)

3) Upper Bound Creation

By using global optimization, create an upper bound on the problem (helpful if branch-and-bound is used)

mp-MIQP Framework – Integer Handling

Integer Handling

1) Choice of Integer Handling

The three techniques known are: (i) the decomposition algorithm (Dua and Pistikopoulos, 2002), (ii) the branch-and-bound algorithm (Oberdieck et al. 2014; Axehill et al., 2014) and (iii) exhaustive enumeration.

2) Fixing the Integer Value

Given a certain integer combination y^*, fix this combination in the mp-MIQP problem and create a corresponding mp-QP problem

mp-MIQP Framework – mp-QP solution

mp-QP solution

1) Handling of left-hand side uncertainty

If uncertainty is present in the constraint matrix, use the approach devised by Wittmann-Hohlbein and Pistikopoulos (2012)

2) Solution of mp-QP problem

The two techniques known are: (i) the geometrical approach (Bemporad et al., 2002; Dua et al., 2002; Tøndel et al., 2003; Spjøtvold et al., 2006), (ii) the combinatorial approach (Gupta et al., 2011; Feller and Johanson, 2013).

3) Determination of valid solution

Given a certain mp-QP solution, classify whether this solution is a valid solution of not. This classification depends on the choice of the integer handling

mp-MIQP Framework – Non-convexity

Non-convexity

Choice of handling non-convexity

Option 1: No comparison procedure, an envelope of solutions is created (Dua et al., 2002; Axehill et al., 2014).

Option 2: The solution and the upper bound are compared, and the resulting non-convexity is enclosed using affine relaxation (Oberdieck et al., 2014).

Option 3: The solution and the upper bound are compared and the resulting non-convexity is taken into account explicitly, resulting in non-convex critical regions. This approach has not been presented in the open literature (ongoing).

mp-MIQP Framework – Termination

Termination

1) Determine parallelization strategy

The solution procedure can be parallelized on different machines. The amount of communication between the main program and the machines determines the autonomy of these threads.

2) Termination criterion

Depending on the integer handling strategy, this criterion varies. However, if it is fulfilled then the algorithm terminates, if not then another iteration is performed

mp-MIQP Framework – A Motivating Example

- **Consider the following example problem**

$$
\begin{aligned}
z(\theta) = \min_{x,y} \;& x_1^2 + 6x_1y_1 + x_2^2 - 20x_2y_2 + y_1^2 + y_2^2 - 5x_1\theta_1 + 20y_1 - 15.5y_2 \\
\text{s.t.} \quad & x_1 - 3y_1 \le 2 + \theta_2 \\
& x_2 + y_1 + 2y_2 \le 1 + \theta_1 \\
& x_1 + x_2 \le 5\theta_1 + 3\theta_2 \\
& \\
& x \in \mathbb{R}^2, \quad x \ge 0, \quad y \in \{0,1\}^2 \\
& \theta \in \Theta = \{\theta \in \mathbb{R}^2 \,|0 \le \theta_l \le 10, l = 1,2\}
\end{aligned}
$$

Matrix of mp-QP problem

$$
Q = \begin{bmatrix} 1 & 0 & 6 & 0 \\ 0 & 1 & 0 & -20 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad Q_x = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}
$$

Full ω matrix

mp-MIQP Framework – Pre-Processing

- **In order to obtain a good upper bound, one iteration of the decomposition algorithm is solved:**

1.

$$
\begin{aligned}
z(\theta) = \min_{x,y,\theta} \;& x_1^2 + 6x_1y_1 + x_2^2 - 20x_2y_2 + y_1^2 + y_2^2 - 5x_1\theta_1 + 20y_1 - 15.5y_2 \\
\text{s.t.} \quad & x_1 - 2y_1 \leq 3 + \theta_2 \\
& x_2 + y_1 + 2y_2 \leq 1 + \theta_1 \\
& x_1 + x_2 \leq 5\theta_1 + 3\theta_2 \\
& \\
& x \in \mathbb{R}^2, \quad x \geq 0, \quad y \in \{0,1\}^2 \\
& \theta \in \Theta = \{\theta \in \mathbb{R}^2 \,|0 \leq \theta_l \leq 10, l = 1,2\}
\end{aligned}
$$

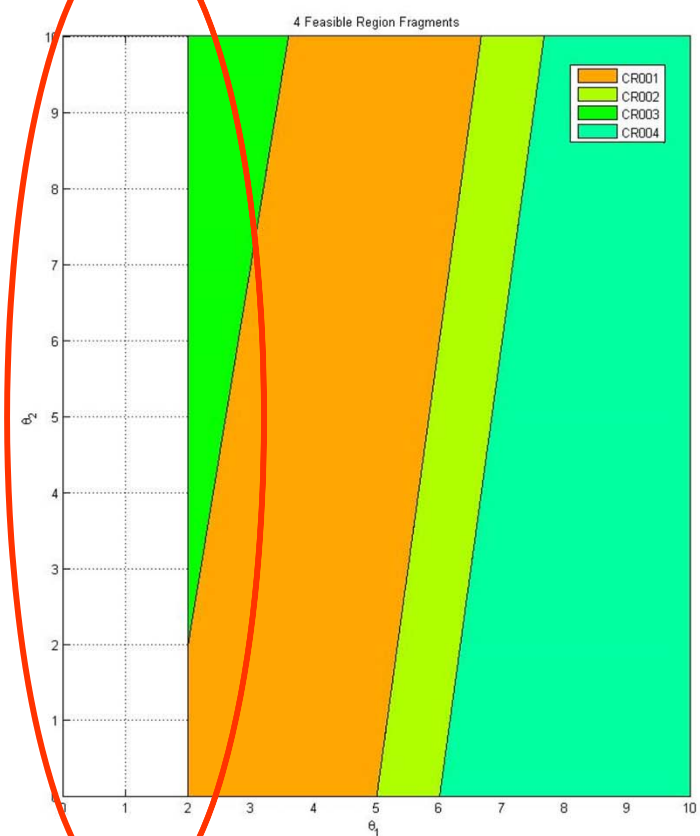
Solve MINLP to global optimality!

2. Retrieve $y^* = [1,1]$ and solve

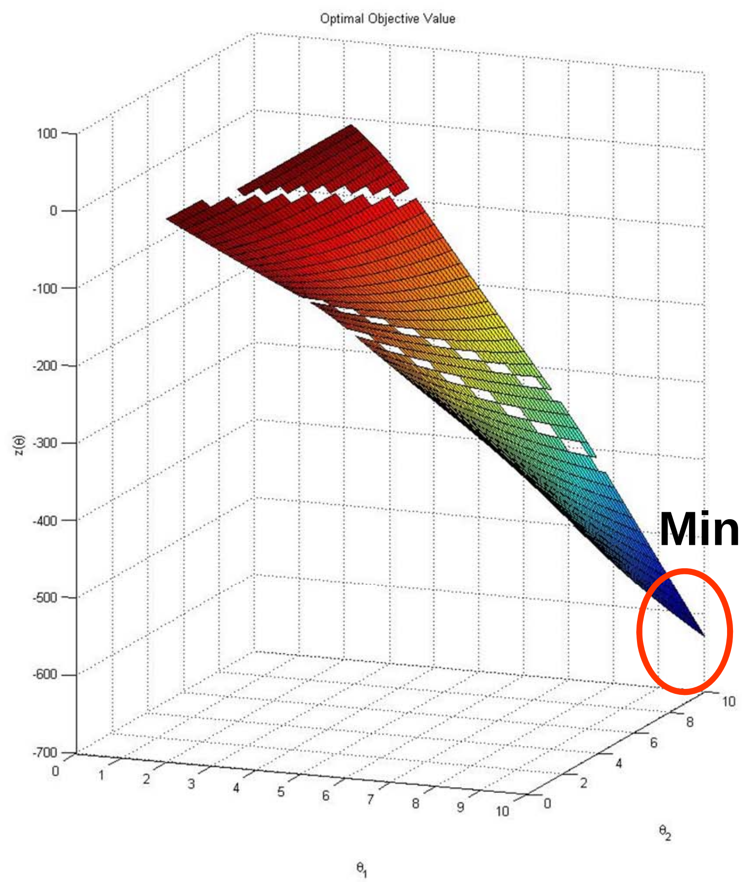
$$
\begin{aligned}
z_{1,1}(\theta) = \min_{x} \;& x_1^2 + x_2^2 - 5x_1\theta_1 + 6x_1 - 20x_2 + 6.5 \\
\text{s.t.} \quad & x_1 \leq 5 + \theta_2 \\
& x_2 \leq -2 + \theta_1 \\
& x_1 + x_2 \leq 5\theta_1 + 3\theta_2 \\
& \\
& x \in \mathbb{R}^2, \quad x \geq 0 \\
& \theta \in \Theta = \{\theta \in \mathbb{R}^2 \,|0 \leq \theta_l \leq 10, l = 1,2\}
\end{aligned}
$$

mp-MIQP Framework – Pre-Processing

- The solution is given by



Infeasibility!



Minimum!

mp-MIQP Framework – Integer Handling

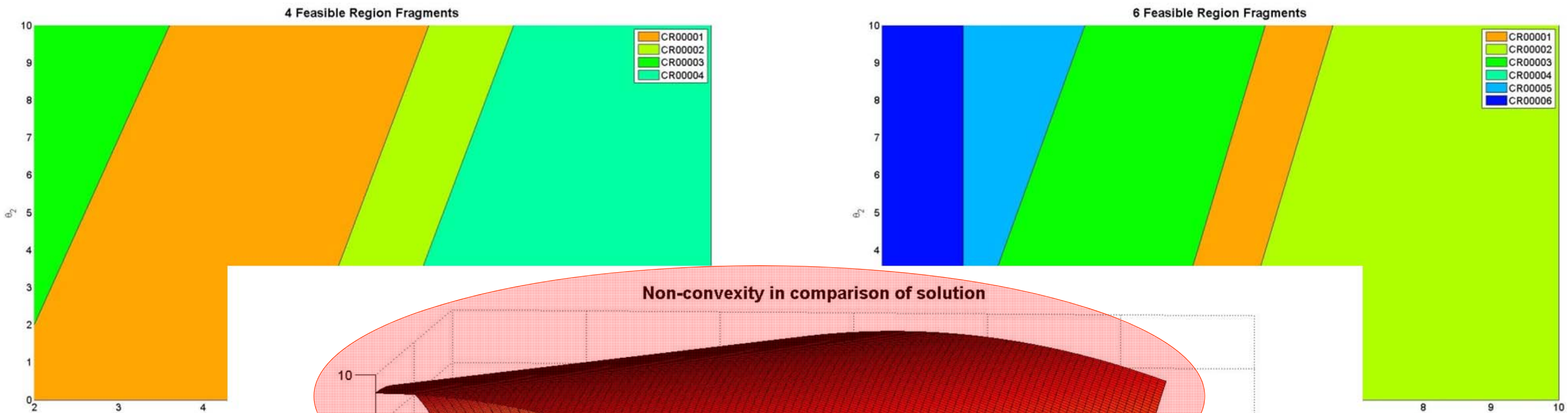
- **The three possible options are**
  1. Decomposition Algorithm (as shown in the Pre-Processing)
  2. Branch-And-Bound
  3. Exhaustive Enumeration

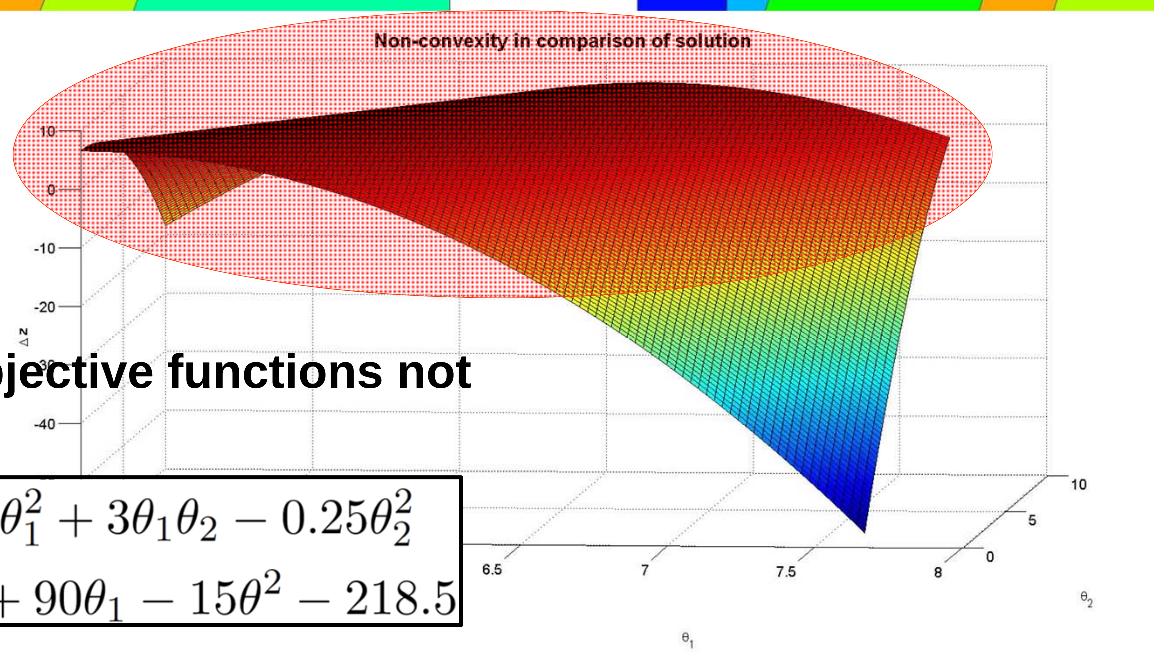
	$y_2 = 0$	$y_2 = 1$
$y_1 = 0$	$z_{0,0}(\theta) = \min_x \; x_1^2 + x_2^2 - 5x_1\theta_1$ s.t. $x_1 \leq 2 + \theta_2$, $x_2 \leq 1 + \theta_1$, $x_1 + x_2 \leq 5\theta_1 + 3\theta_2$	$z_{0,1}(\theta) = \min_x \; x_1^2 + x_2^2 - 5x_1\theta_1 - 20x_2 - 14.5$ s.t. $x_1 \leq 2 + \theta_2$, $x_2 \leq -1 + \theta_1$, $x_1 + x_2 \leq 5\theta_1 + 3\theta_2$
$y_1 = 1$	$z_{1,0}(\theta) = \min_x \; x_1^2 + x_2^2 - 5x_1\theta_1 + 6x_1 + 21$ s.t. $x_1 \leq 5 + \theta_2$, $x_2 \leq \theta_1$, $x_1 + x_2 \leq 5\theta_1 + 3\theta_2$	$z_{1,1}(\theta) = \min_x \; x_1^2 + x_2^2 - 5x_1\theta_1 + 6x_1 - 20x_2 + 6.5$ s.t. $x_1 \leq 5 + \theta_2$, $x_2 \leq -2 + \theta_1$, $x_1 + x_2 \leq 5\theta_1 + 3\theta_2$

mp-MIQP Framework – Non-convexity

- **Consider the two solutions**

$$\boldsymbol{y} = [\mathbf{1}; \mathbf{1}] \qquad\qquad \boldsymbol{y} = [\mathbf{0}; \mathbf{0}]$$



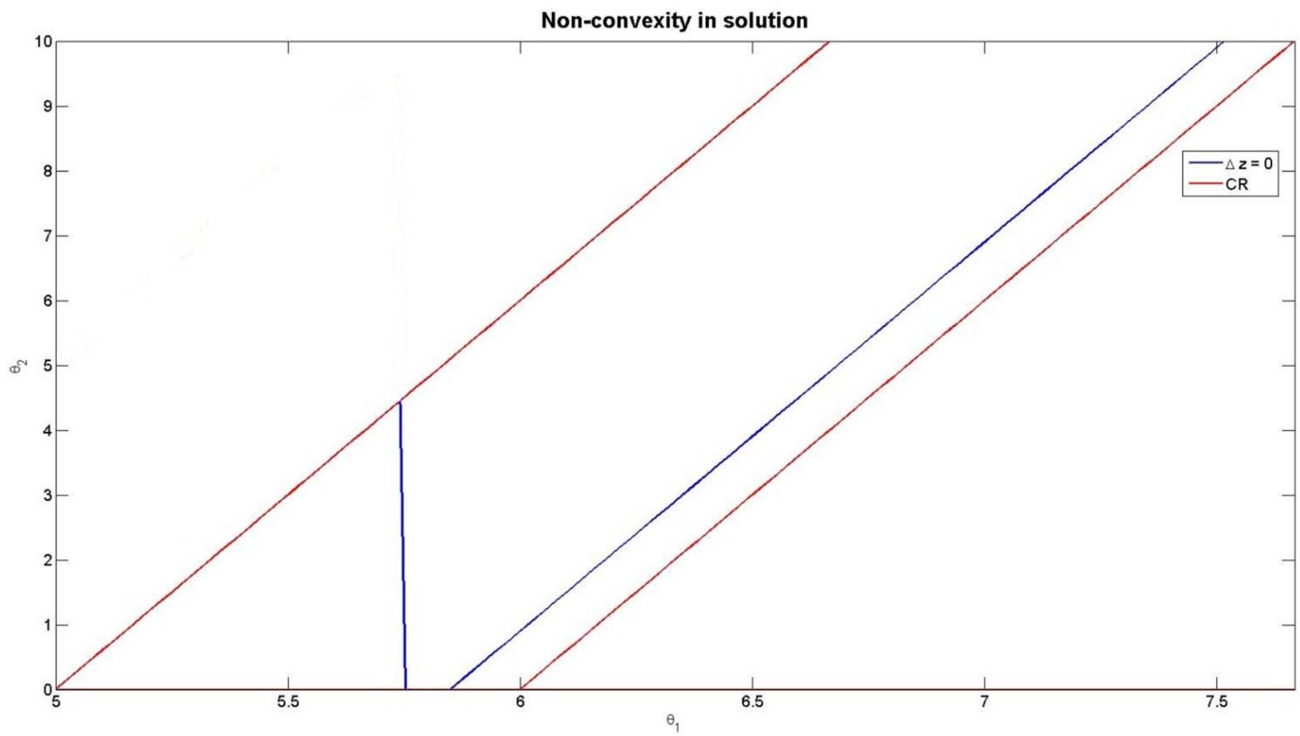


Difference of objective functions not affine!

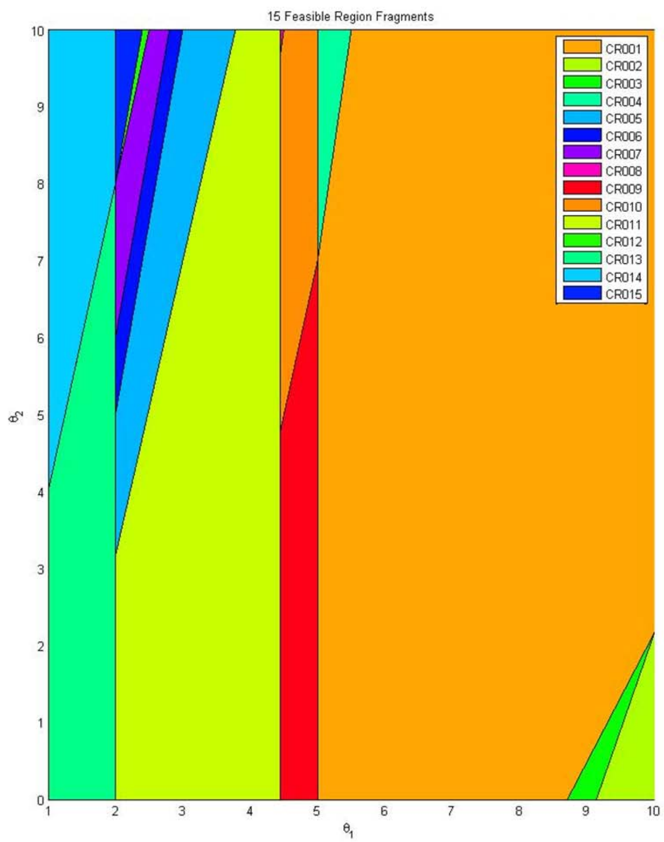
$$\Delta z = -9\theta_1^2 + 3\theta_1\theta_2 - 0.25\theta_2^2 + 90\theta_1 - 15\theta^2 - 218.5$$

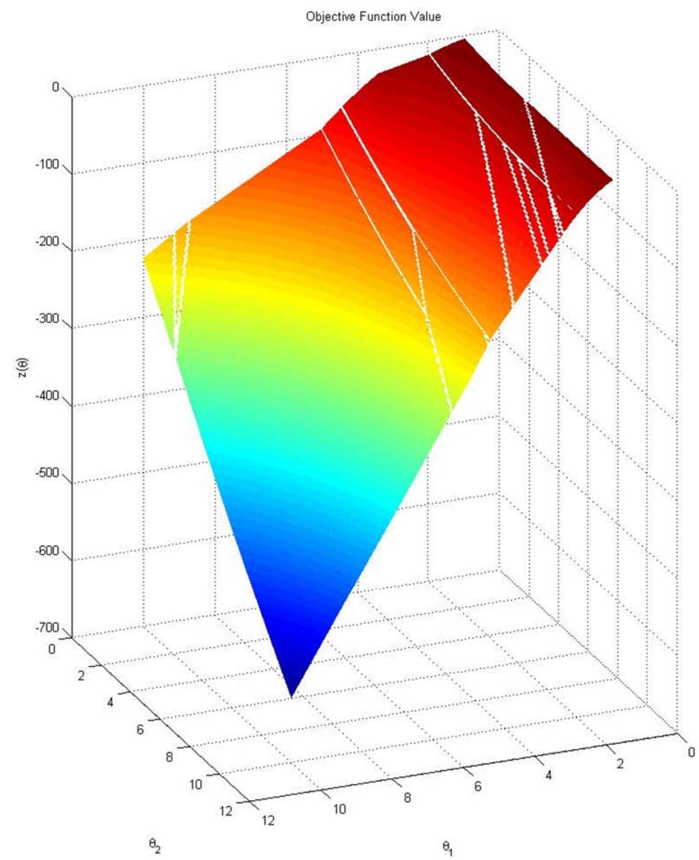
mp-MIQP Framework – Non-convexity

- **Two options:**
  1. Envelope of Solutions (accumulate the solutions)
  2. Create affine relaxation of $\Delta z = 0$ (e.g. McCormick relaxation, McCormick (1976)), and proceed accordingly



mp-MIQP Framework – Example Solution





mp-MIQP Framework – Comments

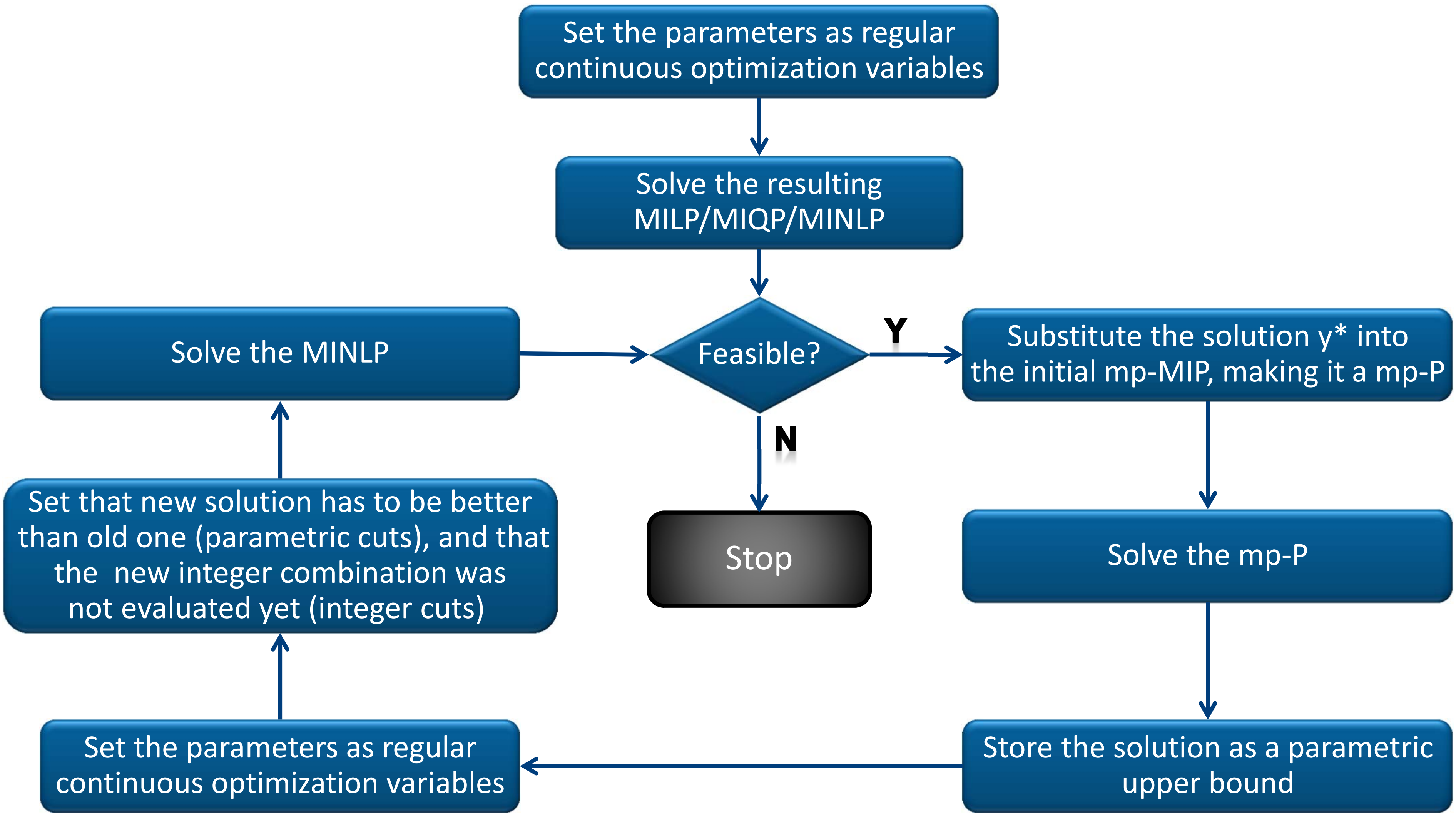
- **This framework captures all developments in the area of mp-MIQP problems so far**

- **Any combination of the options inside one box results in an alternative mp-MIQP algorithm!**

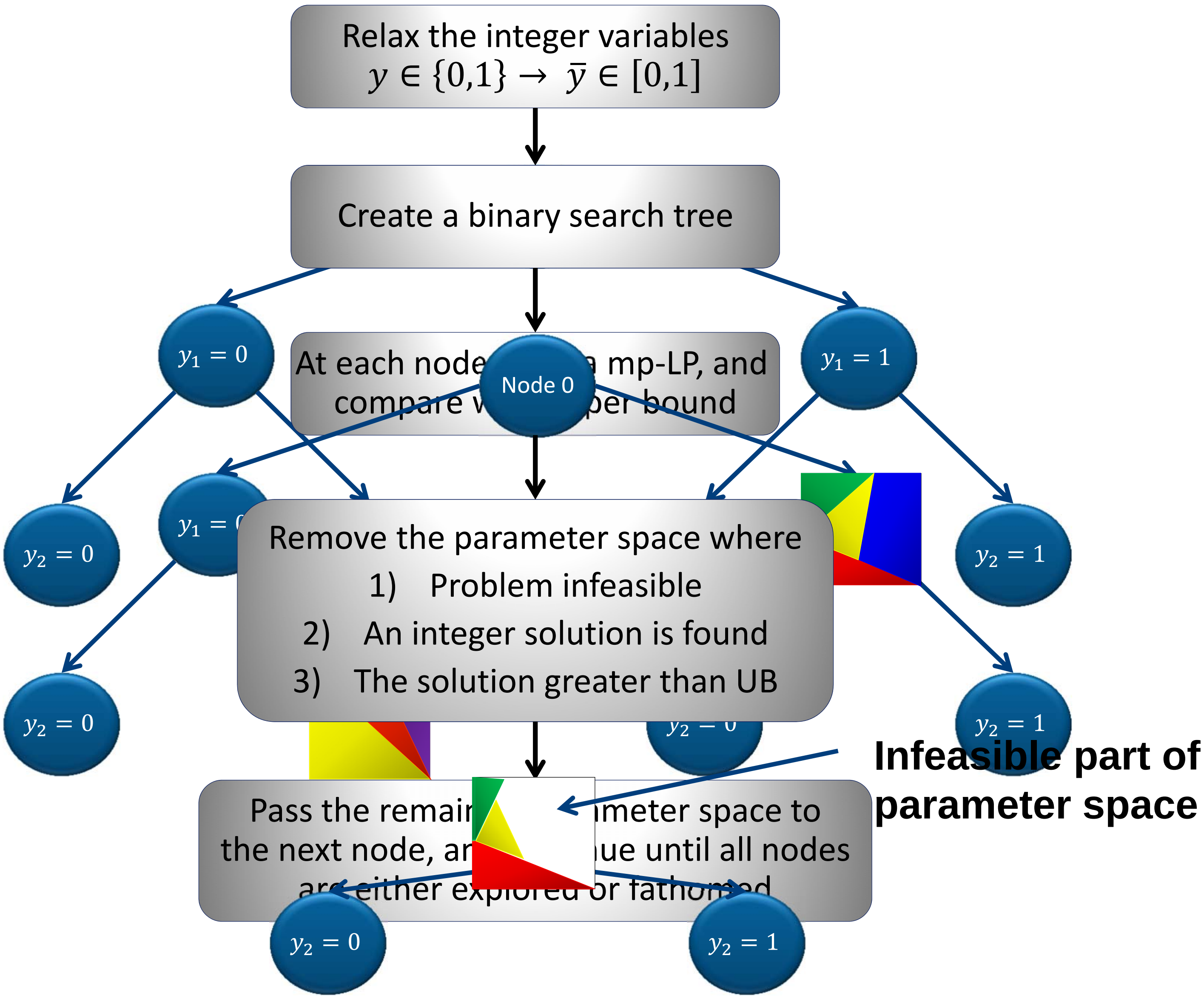
- **As long as there is no new way of handling integer variables, all mp-MIQP algorithms will follow this structure!**

- **Let us have a closer look at the two extreme cases: the decomposition algorithm and the branch-and-bound algorithm**

mp-MIQP Framework – Decomposition Algorithm



mp-MIQP Framework – Branch-And-Bound Algorithm



mp-MIQP Framework – Parallelization

The main algorithm consists of an iterative procedure

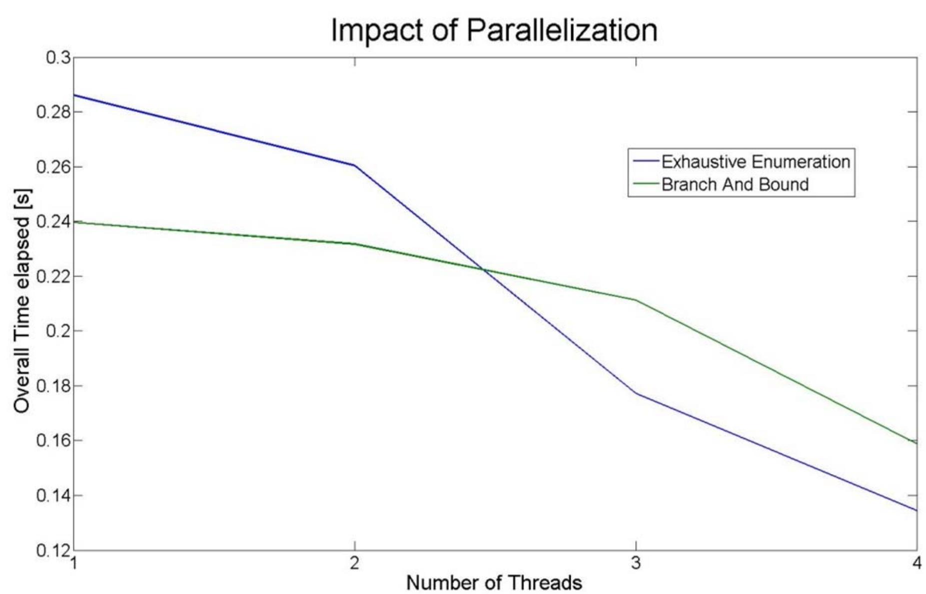
Solve the same problem over and over!

Parallelization!

N Problems on 1 thread → **N/k problems on k threads**

mp-MIQP Framework – Preliminary Results Parallelization

First results indicate a reduction of computational time of 50 – 60% when parallelized on 4 threads



mp-MIQP problems – Applications

- **Scheduling under Uncertainty**
- **Integration of Scheduling and mp-MPC**
- **Robust/nominal hybrid MPC**

Application 1 – Scheduling under Uncertainty (Wittmann-Hohlbein and Pistikopoulos, 2013)

1. Classification of Uncertainty

Identify and group parameters according to their availability at decision stage:

- Revealing: Their value is known
- Non-revealing: Value is not known

2. Approximation Stage

A) Immunize the problem against uncertainty
   - in the constraint matrix A attributed to the revealing parameters
   - in all coefficients attributed to non-revealing parameters

B) Transform the problem into its robust counterpart, resulting in a multiparametric MILP

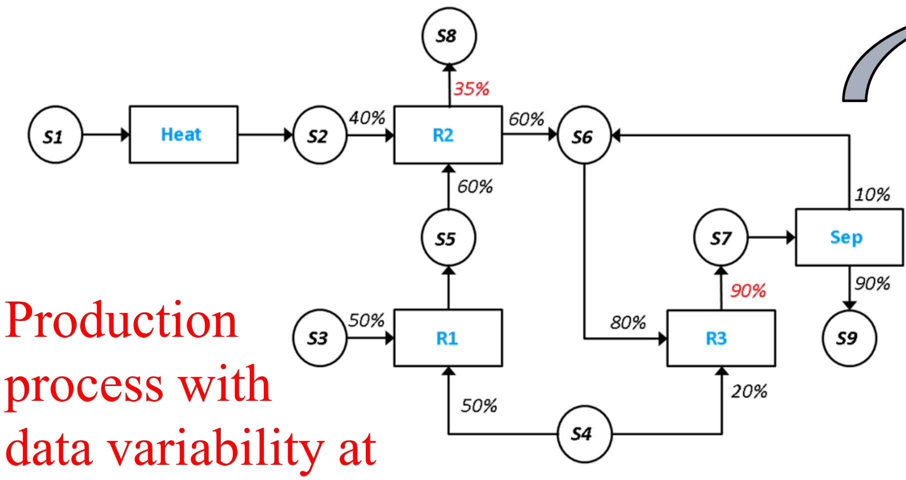
3. Multiparametric Programming

Solve the mp-MILP using one of the available solvers

$$
\begin{aligned}
z(\theta) = \min_{\omega} & \left(P^T\theta + c\right)^T \omega + d^T\theta \\
\text{s.t.} \quad & A\omega \le b + F\theta \\
& \omega \in \Omega = \mathbb{R}^n \times \{0,1\}^m \\
& \theta \in \Theta = \{\theta \in \mathbb{R}^q \,|\theta_l^{min} \le \theta_l \le \theta_l^{max}, l = 1, \ldots, q\}
\end{aligned}
$$

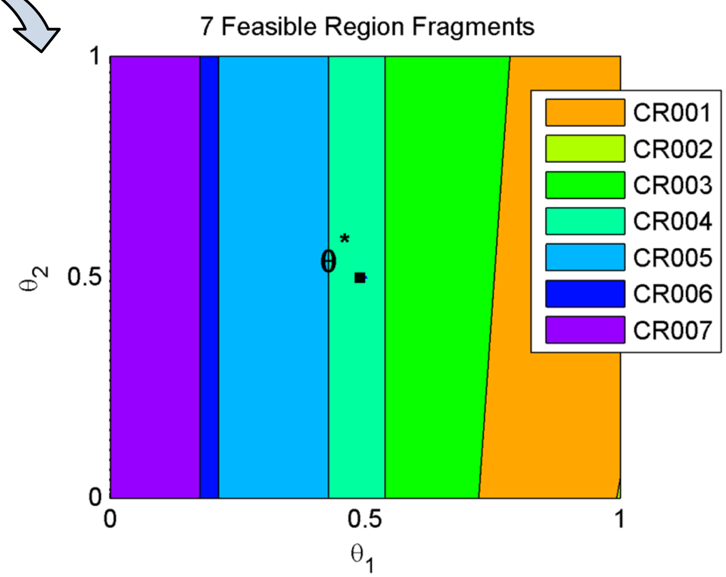
A scheduling problem under uncertainty reduces to the solution of a multiparametric MILP/MIQP

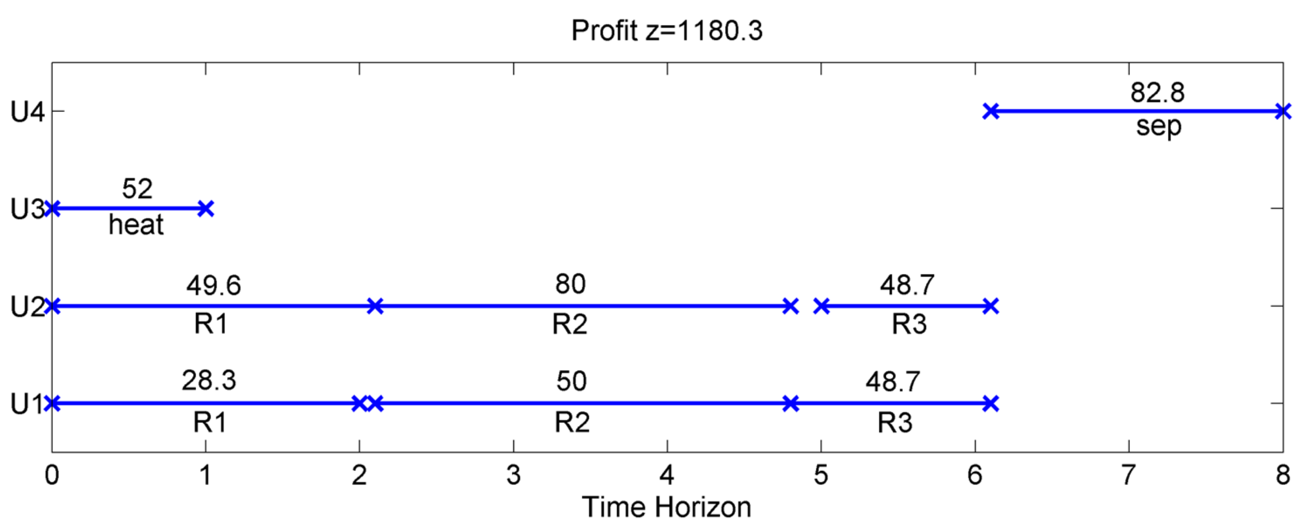
Application 1 – Scheduling under Uncertainty (Wittmann-Hohlbein and Pistikopoulos, 2013)



Production process with data variability at optimization stage:
mp-MILP model

Explicit solution of partially robust scheduling model





Function Evaluation:
Approximate scheduling policy for realized θ*

Application 2 – Integration of Scheduling and mp-MPC (Kopanos and Pistikopoulos, 2014)

Step 0: DERIVE A STATE-SPACE SCHEDULING MODEL

Identify the inputs, the state variables, and the disturbances of the system. And then, reformulate the original scheduling model to a state-space scheduling model.

Step 1: FORMULATE A MULTIPARAMETRIC PROGRAMMING STATE-SPACE SCHEDULING MODEL

Use the state-space scheduling model, and consider as bounded uncertain parameters the disturbances & the initial state variables.

Step 2: SOLVE THE RESULTING MULTIPARAMETRIC PROGRAMMING (MP) PROBLEM [OFF-LINE & ONCE]

Define a prediction horizon & solve the resulting multiparametric problem. Then, save the critical regions & the corresponding variables functions for the whole range of uncertain parameters.

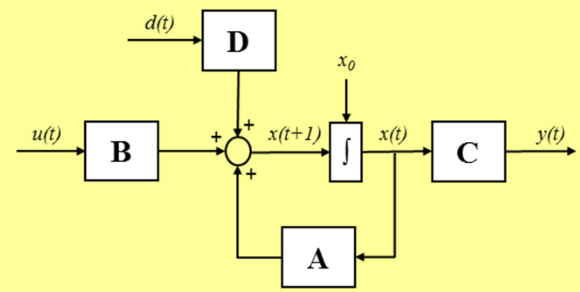
Step 3: REACTIVE SCHEDULING VIA A MP-BASED ROLLING HORIZON APPROACH

Define a control horizon, and use the output of Step 2 within a rolling horizon scheme for the iterative reactive scheduling.

Reactive scheduling via multiparametric programming rolling horizon.

State-space representation for a linear discrete-time system:

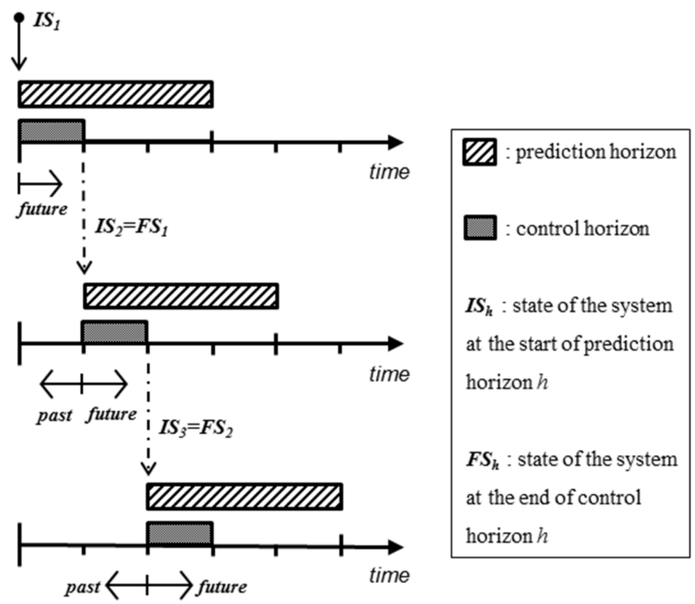
$$x(t+1) = A(t)x(t) + B(t)u(t) + D(t)d(t)$$
$$x(t=0) = x_0$$
$$y(t) = C(t)x(t)$$



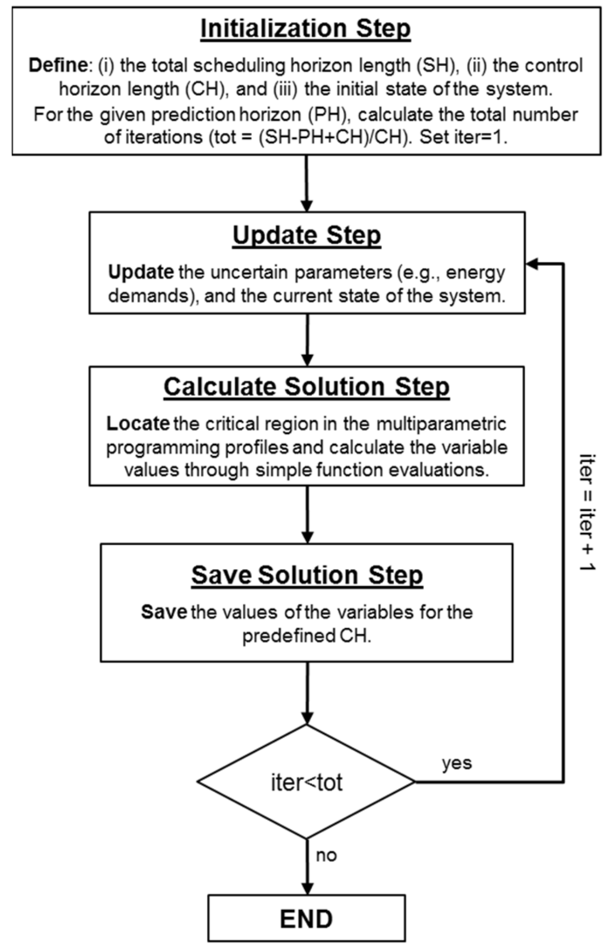
Block diagram for the state-space representation of linear discrete-time systems.

$$\text{mpMILP-RHS}\begin{cases} z(\theta) := \min\limits_{x,y}(c^T x + d^T y) \\ \text{subject to} \\ Ax + Ey \leq b + F\theta \\ x \in \mathbb{R}^n,\ y \in \{0,1\}^p \\ \theta \in \Theta := \{\theta \in \mathbb{R}^q \mid \theta_l^{min} \leq \theta_l \leq \theta_l^{max}, l = 1, \ldots, q\} \end{cases}$$

Application 2 – Integration of Scheduling and mp-MPC (Kopanos and Pistikopoulos, 2014)



Reactive scheduling via a rolling horizon framework.



An algorithm for rolling horizon via multiparametric programming.

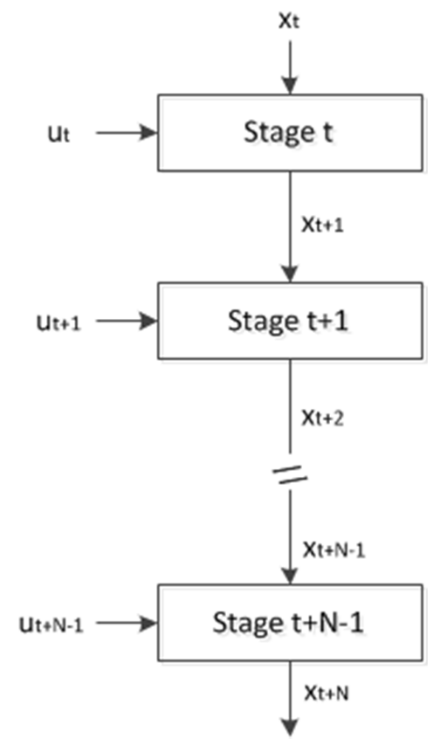
Note:
- *The initial state of the system in the actual prediction horizon is equal to the final state of the system in the previous control horizon.*
- *The **system receives feedback** (e.g., actual demand, updated state of system) **at every discrete time instant**.*

Application 3 – Hybrid Control (Rivotti and Pistikopoulos, 2014; Oberdieck and Pistikopoulos, 2014)

Two-stage approach for optimal/robust hybrid control

1) **Use Dynamic Programming to decompose the problem into smaller subproblems**
   - Each subproblem only solves for the current stage *k*
   - Treat states and future control actions as parameters

2) **Solve the resulting mp-MILP/ mp-MIQP for each stage!**



Conclusion – Outlook and Future Research

- **In a nutshell**
  - A framework for the general solution of mp-MIQP problems
  - Applicability to a wide class of engineering problems
- **Ongoing**

  A. Research towards the exact solution of mp-MIQP problems

  B. Complete software implementation of the framework

  C. Applications

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