# **Easy** and **not so easy** multifacility location problems... *(In 20 minutes.)*

MINLP 2014
Pittsburgh, June 2014

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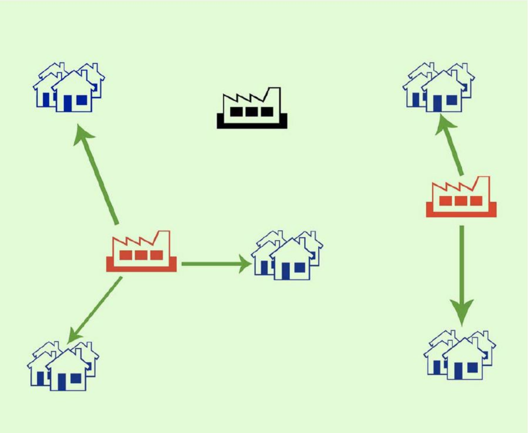
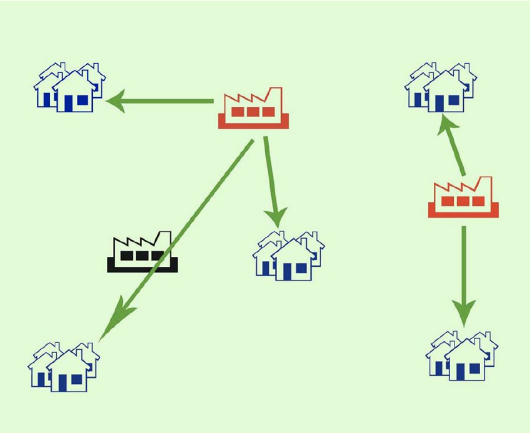
# Outline

1. Introduction (In 3 minutes)
2. Discrete ordered median problem (in 7 minutes)
3. Continuous multifacility problems (10 minutes)
4. Dimensionality reduction of the SDP-rel for multifacility location problems (in no time)

# A few minutes of motivation

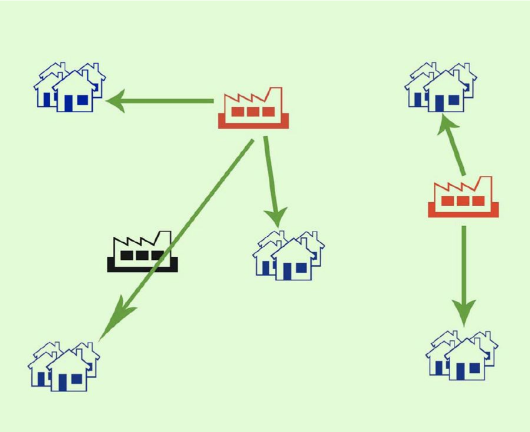


# A few minutes of motivation



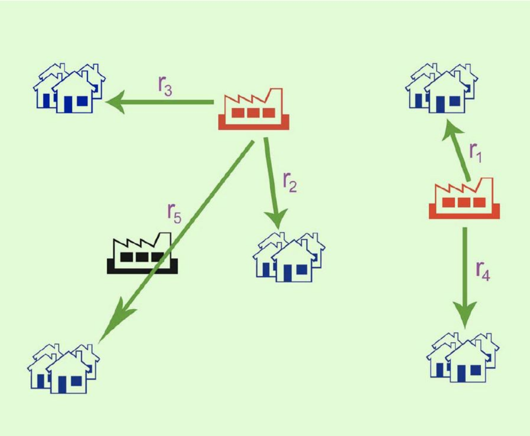
Logistic problem or in another jargon *Clustering or Classification* Problem

# A few minutes of motivation

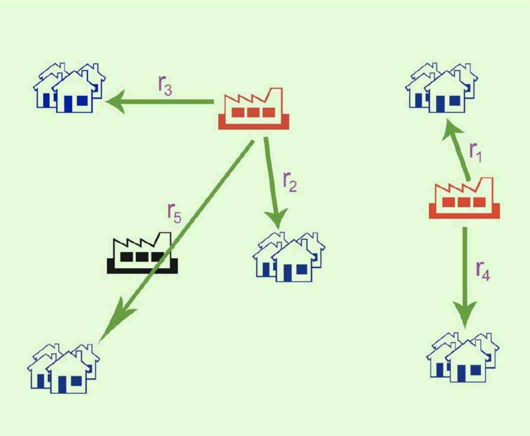


Logistic problem or in another jargon *Clustering or Classification* Problem

# Discrete ordered median problem: Modeling framework



# Discrete ordered median problem: Modeling framework




$$r_1 \leq r_2 \leq r_3 \leq r_4 \leq r_5$$

$$\lambda_1 r_1 + \lambda_2 r_2 + \lambda_3 r_3 + \lambda_4 r_4 + \lambda_5 r_5$$

# Ordered Median Functions

**Definition**

Let $x \in \mathbb{R}^n$ and $\lambda \in \mathbb{R}^n$. The ordered median function, $f_\lambda$ is defined as:

$$f_\lambda(x) = \langle \lambda, \mathrm{sort}(x) \rangle$$

where $\mathrm{sort}(x) = (x_{(1)}, \ldots, x_{(n)})$ with $x_{(i)} \in \{x_1, \ldots, x_n\}$ and such that $x_{(1)} \leq x_{(2)} \leq \cdots \leq x_{(n)}$.

# Special Cases

| | | |
|---|---|---|
| Mean | $(\frac{1}{n}, \cdots, \frac{1}{n})$ | $\frac{1}{n}\sum_{i=1}^{n} x_i$ |
| Minimum | $(1, 0, \cdots, 0)$ | $\min_{1\leq i\leq n} x_i$ |
| Maximum | $(0, 0, \cdots, 1)$ | $\max_{1\leq i\leq n} x_i$ |
| $k$-centrum | $(0, \cdots, 0, \overbrace{1, \cdots, 1}^{k})$ | $\sum_{i=n-k+1}^{n} x_{(i)}$ |
| anti-$k$-centrum | $(\overbrace{1, \cdots, 1}^{k}, 0, \cdots, 0)$ | $\sum_{i=1}^{k} x_{(i)}$ |
| $(k_1, k_2)$-Trimmed mean | $(\overbrace{0, \cdots, 0}^{k_1}, 1, \cdots, 1, \overbrace{0, \cdots, 0}^{k_2})$ | $\sum_{i=k_1+1}^{n-k_2} x_{(i)}$ |
| Range | $(-1, 0, \cdots, 0, 1)$ | $\max_{1\leq i\leq n} x_i - \min_{1\leq i\leq n} x_i$ |
| ... | ... | ... |

# Our program starts about 2006 ...

# Previous Works

- Discrete Case: (Nickel & P., 2005; Boland, Marin, Nickel, P., 2006; Marín, Nickel, P., 2009).
- Networks: ( Nickel & P., 1999; Kalcsics, Nickel, P., 2002; Kalcsics, Nickel, P., Tamir, 2006; P. & Tamir, 2005);
- Continuous:
  - Planar Weber Problem: (Weiszfeld, 1937; and extensions...)
  - Planar Euclidean: BTST (Drezner, 2007); D.C. (Drezner& Nickel, 2009).
  - Planar Convex OM with $\ell_p$ norms: (Espejo, et. al, 2009).
  - Planar Euclidean $k$-centrum: (Rodríguez-Chía et. al, 2010).
  - OM 1-Facility Location, any dimension, any $\ell_p$ or polyhedral norm: (Blanco, P. & ElHaj-BenAli, 2013, 2014).

# 7 minutes for the DOMP: An INLP

- $A = \{a_1, \ldots, a_n\}$
- $C = (c_{ij})_{i,j=1,\ldots,n}$
- $X \subseteq A$ with $|X| = p \leq p$
- $c_i(X) := \min_{k \in X} c_{ik}$
- $\sigma_X$ a permutation of $\{1, \ldots, n\}$

$$c_{\sigma_X(1)}(X) \leq c_{\sigma_X(2)}(X) \leq \cdots \leq c_{\sigma_X(n)}(X)$$

## Discrete ordered median problem (DOpP)

$$\min_{X \subseteq A\,,\ |X|=p} \sum_{i=1}^{n} \lambda_i c_{\sigma_X(i)}(X)\ .$$

with $\lambda = (\lambda_1, \ldots, \lambda_n)$ y $\lambda_i \geq 0,\ i = 1, \ldots, n.$

# Sorting as an integer program (IP)

- **INPUT:** real numbers $r_1, \ldots, r_n$.
- **OUTPUT:** $r_{\sigma(1)} \leq r_{\sigma(2)} \leq \ldots \leq r_{\sigma(n)}$, $\sigma \in \Pi(\{1, \ldots, n\})$.
- **Decision variables:**

$$s_{ki} = \begin{cases} 1 & \text{if } \sigma(k) = i \\ 0 & \text{otherwise} \end{cases} \qquad \forall i, k = 1, \ldots, n.$$

$$r_{\sigma(k)} := \sum_{i=1}^{n} s_{ki} r_i, \; k = 1, \ldots, n$$

Desired output obtained

## Sorting as an IP:

(**SORT**) minimize 1

s.t.

$$\sum_{k=1}^{n} s_{ki} = 1 \qquad \forall i = 1, \ldots, n$$

$$\sum_{i=1}^{n} s_{ki} = 1 \qquad \forall k = 1, \ldots, n$$

$$\sum_{i=1}^{n} s_{ki} r_i \leq \sum_{i=1}^{n} s_{k+1,i} r_i \qquad \forall k = 1, \ldots, n-1$$

$$s_{ki} \in \{0, 1\} \qquad \forall i, k = 1, \ldots, n.$$

# p-median formulation

## Decision variables:

$$y_j = \begin{cases} 1 & \text{a new facility is built in } a_j \\ 0 & \text{otherwise} \end{cases} \quad \text{and } x_{ij} = \begin{cases} 1 & \text{site } a_i \text{ is served by facility } a_j \\ 0 & \text{otherwise} \end{cases}$$

$$\text{minimize} \quad \sum_{i=1}^{n} \sum_{j=1}^{n} c_{ij} \, x_{ij}$$

$$\sum_{j=1}^{n} y_j = p$$

$$\sum_{j=1}^{n} x_{ij} = 1 \qquad \forall i = 1, \ldots, n$$

$$y_j \geq x_{ij} \qquad \forall j, i = 1, \ldots, n$$

$$y_j \in \{0, 1\}, \quad x_{ij} \geq 0 \qquad \forall j, i = 1, \ldots, n$$

# Quadratic formulation for DOMP: Constraints to sort

Setting $r_i = \sum_{j=1}^{n} c_{ij}x_{ij}, \ \forall i = 1, \dots, n.$ (SORT+ $p$-median) (**DOMP**)

minimize $\quad \sum_{k=1}^{n} \lambda_k \sum_{i=1}^{n} s_{ki} \sum_{j=1}^{n} c_{ij}x_{ij}$

$$\sum_{k=1}^{n} s_{ki} = 1 \qquad \forall i = 1, \dots, n$$

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$$\sum_{i=1}^{n} s_{ki} \sum_{j=1}^{n} c_{ij}x_{ij} \leq \sum_{i=1}^{n} s_{k+1,i} \sum_{j=1}^{n} c_{ij}x_{ij} \qquad \forall k = 1, \dots, n-1$$

# Quadratic formulation for DOMP: Constraints to sort

## Constraints corresponding to p-median problem

$$\sum_{j=1}^{n} y_j = p$$

$$\sum_{j=1}^{n} x_{ij} = 1 \qquad \forall i = 1, \dots, n$$

$$y_j \geq x_{ij} \qquad \forall j, i = 1, \dots, n$$

$$x_{ij} \geq 0 \qquad \forall j, i = 1, \dots, n$$

$$s_{ki} \in \{0, 1\} \qquad \forall i, k = 1, \dots, n$$

$$y_j \in \{0, 1\} \qquad \forall j = 1, \dots, n$$

# A first linearization: DOMP1

Constraints to sort. Adding artificial variables $x_{ij}^k$

**(L1)** minimize $\sum_{k=1}^n \lambda_k \sum_{i=1}^n \sum_{j=1}^n c_{ij} \underbrace{s_{ki} x_{ij}}_{x_{ij}^k}$

$$\sum_{k=1}^{n} \underbrace{s_{ki}}_{\sum_j x_{ij}^k} = 1 \qquad \forall k = 1, \ldots, n$$

$$\sum_{i=1}^{n} \underbrace{s_{ki}}_{\sum_j x_{ij}^k} = 1 \qquad \forall k = 1, \ldots, n$$

$$\sum_{i=1}^{n} \sum_{j=1}^{n} c_{ij} \underbrace{s_{ki} x_{ij}}_{x_{ij}^k} \leq \sum_{i=1}^{n} \sum_{j=1}^{n} c_{ij} \underbrace{s_{k+1,i} x_{ij}}_{x_{ij}^{k+1}} \qquad \forall k = 1, \ldots, n-1$$

# Linearization

## Constraints corresponding to $p$-median problem

$$\sum_j y_j = p$$

$$\left( \sum_j \underbrace{x_{ij}}_{\sum_k x_{ij}^k} = 1 \qquad \forall i = 1, \ldots, n \right)$$

$$y_j \geq \underbrace{x_{ij}}_{\sum_k x_{ij}^k} \qquad \forall j, i = 1, \ldots, n$$

$$x_{ij}^k \in \{0, 1\} \qquad \forall i, k, j = 1, \ldots, n$$

$$y_j \in \{0, 1\} \qquad \forall j = 1, \ldots, n$$

# A new formulation for DOMP: DOMP3

$$\sum_{i=1}^{n}\sum_{j=1}^{n} c_{ij} x_{ij}^{k} \leq \sum_{i=1}^{n}\sum_{j=1}^{n} c_{ij} x_{ij}^{k+1} \quad \forall k = 1, \ldots, n-1$$

Example

$$C = \begin{pmatrix} 0 & 2 & 7 & 4 \\ 1 & 0 & 5 & 5 \\ 3 & 6 & 0 & 2 \\ 9 & 4 & 1 & 0 \end{pmatrix}$$

$$11 \prec 22 \prec 33 \prec 44 \prec 21 \prec 43 \prec 12 \prec 34 \prec$$

$$31 \prec 14 \prec 42 \prec 23 \prec 24 \prec 32 \prec 13 \prec 41$$

# A new formulation for DOMP: DOMP3

$$\sum_{i=1}^{n}\sum_{j=1}^{n} c_{ij}x_{ij}^{k} \leq \sum_{i=1}^{n}\sum_{j=1}^{n} c_{ij}x_{ij}^{k+1} \quad \forall k = 1,\dots,n-1$$

## Example

$$C = \begin{pmatrix} 0 & 2 & 7 & 4 \\ 1 & 0 & 5 & 5 \\ 3 & 6 & 0 & 2 \\ 9 & 4 & 1 & 0 \end{pmatrix}$$

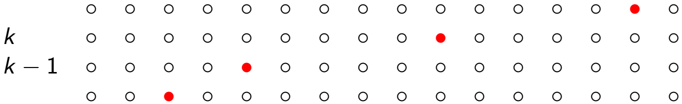
$$11 \prec 22 \prec 33 \prec 44 \prec 21 \prec 43 \prec 12 \prec 34 \prec$$

$$31 \prec 14 \prec 42 \prec 23 \prec 24 \prec 32 \prec 13 \prec 41$$

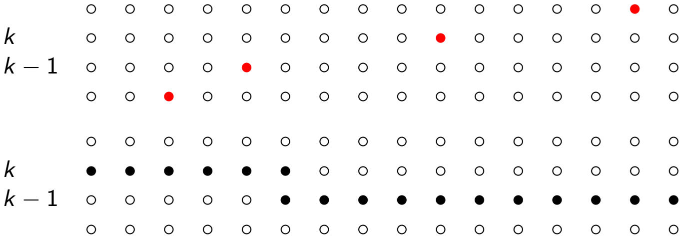
$$x_{ij}^{k} = \begin{cases} 1 & \text{if facility } i \text{ allocated to server } j \text{ and} \\ & \quad C_{ij} \text{ is the } k\text{-th smallest cost} \\ 0 & \text{otherwise} \end{cases}$$

$$y_j = \begin{cases} 1 & \text{if } j \in J \\ 0 & \text{if } j \notin J \end{cases}$$

# The rationale: Incompatibilities



# The rationale: Incompatibilities



# $DOMP_3$

$$
\begin{aligned}
\min \quad & \sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{k=1}^{n} \lambda^k C_{ij} x_{ij}^k \\
s.t. \quad & \sum_{j=1}^{n}\sum_{k=1}^{n} x_{ij}^k = 1 && \forall i = 1, \dots, n \\
& \sum_{i=1}^{n}\sum_{j=1}^{n} x_{ij}^k = 1 && \forall k = 2, \dots, n \\
& \sum_{k=1}^{n} x_{ij}^k \leq y_j && \forall i, j = 1, \dots, n \\
& \sum_{j=1}^{n} y_j = p \\
& \sum_{\hat{\imath}\hat{\jmath} \succeq ij} x_{\hat{\imath}\hat{\jmath}}^{k-1} + \sum_{\hat{\imath}\hat{\jmath} \preceq ij} x_{\hat{\imath}\hat{\jmath}}^{k} \leq 1 && \forall i, j = 1, \cdots, n \forall k = 2, \cdots, n \\
& x_{ij}^k \in \{0, 1\} && \forall i, j, k = 1, \dots, n
\end{aligned}
$$

# $DOMP_4$ **Formulation**

$$
\begin{aligned}
\min \quad & \sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{k=1}^{n} \lambda^k C_{ij} x_{ij}^k \\
s.t. \quad & \sum_{j=1}^{n}\sum_{k=1}^{n} x_{ij}^k = 1 && \forall i = 1,\dots,n \\
& \sum_{i=1}^{n}\sum_{j=1}^{n} x_{ij}^k = 1 && \forall k = 2,\dots,n \\
& \sum_{k=1}^{n} x_{ij}^k \le y_j && \forall i,j = 1,\dots,n \\
& \sum_{j=1}^{n} y_j = p \\
& \sum_{i}^{n}\sum_{j}^{n}\left( \sum_{\substack{i'=1 \\ :i'j'}}^{n}\sum_{\substack{j'=1 \\ \preceq ij}}^{n} x_{i'j'}^{k} + \sum_{\substack{i'=1 \\ :i'j'}}^{n}\sum_{\substack{j'=1 \\ \succeq ij}}^{n} x_{i'j'}^{k-1} \right) \le n^2 && \forall k = 2,\cdots,n \\
& x_{ij}^k \in \{0,1\} && \forall i,j,k = 1,\dots,n
\end{aligned}
$$

# Comparing formulations

## Notation

We denote by $P_l$ the polytope defining the feasible set of the linear relaxation of formulation $DOMP_l$, by $z_l(\cdot)$ the value of the objective function of $DOMP_l$ evaluated at the point $(\cdot)$ and by $P_l^I$ the convex hull of the integer solutions within that polytope.

## Theorem

1. $g(P_2) \subset P_1$.
2. $f(P_3) \subset P_2$.
3. $P_3 \subseteq P_1$.
4. $z_3^{LP} \geq z_2^{LP} \geq z_1^{LP}$.

| Formulation | GAP |
|---|---|
| $DOMP_1$ | 2.79% |
| $DOMP_2$ | 6.18% |
| $DOMP_3$ | 0.92% |
| $DOMP_4$ | 2.57% |
| $DOMP_{4\cap 1}$ | 2.48% |

Table: Average integrality gaps

| Formulation | GAP |
| --- | --- |
| $DOMP_1$ | 2.79% |
| $DOMP_2$ | 6.18% |
| $DOMP_3$ | 0.92% |
| $DOMP_4$ | 2.57% |
| $DOMP_{4\cap 1}$ | 2.48% |

Table: Average integrality gaps

| Solution approach | Time (s) | #nodes |
| --- | --- | --- |
| $DOMP_3(B\&B)$ | 463.24(8) | 46.20 |
| $DOMP_{4\cap 1}(B\&B)$ | 18.64 | 1389.51 |
| $DOMP_{4\cap 1}(B\&C-3)$ | 52.25 | 24.94 |
| $DOMP_4(B\&C-3)$ | 39.39 | 29.37 |

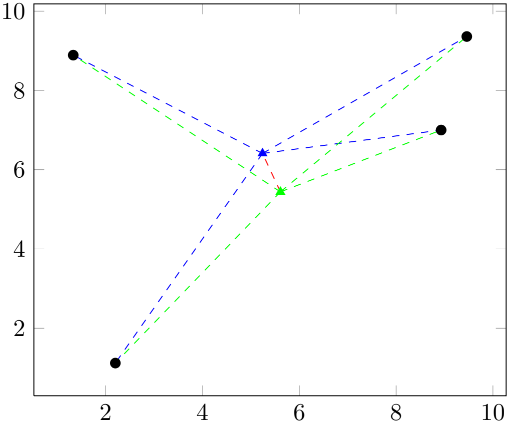
Table: CPU-Time and Number of nodes of the different formulations for $\mathbf{n} = \mathbf{20}, \mathbf{30}, \mathbf{50}$.

# Continuous Multifacility Ordered Median Problems in 10 minutes

| | | |
|---|---|---|
| Demand points | : | $A = \{a_1, \dots, a_n\} \subset \mathbb{R}^d$ |
| Norm | : | $\ell_\tau$ $\left(\|x\|_\tau = \left(\sum_{i=1}^{d} \|x_i\|^\tau\right)^{\frac{1}{\tau}}\right.$, for all $x \in \mathbb{R}^d$) |
| Feasible Domain | : | $\mathbb{R}^d$ |
| Facilities | : | $p$ facilities to be located. |
| Goal | : | Find $p$ points $x_1^*, \dots, x_p^* \in \mathbb{R}^d$ minimizing some globalizing function of the distances to the set $A$. |

# An easy model: Multiple allocation MF OMP

Rodríguez-Chía, Nickel, P., and Fernández. (2000).



# Multiple allocation MF OMP

$$f_\lambda^{NI}(x_1, x_2, \ldots, x_p) = \sum_{i=1}^{n}\sum_{j=1}^{p} \lambda_{ij} d_{(i)}(x_j) + \sum_{j=1}^{p-1}\sum_{j'=j+1}^{p} \mu_{jj'} \|x_j - x_{j'}\|_\tau, \qquad (1)$$

where,

$$X = \{x_1, \ldots, x_p\},$$

$$d_i(x) = \|a_i - x\|_\tau,$$

$$d_{(1)}(x) \geq d_{(2)}(x) \geq \ldots \geq d_{(n)}(x),$$

$$\begin{gathered} \lambda_{11} \geq \lambda_{21} \geq \ldots \geq \lambda_{n1} \geq 0, \\ \lambda_{12} \geq \lambda_{22} \geq \ldots \geq \lambda_{n2} \geq 0, \\ \ldots \\ \lambda_{1p} \geq \lambda_{2p} \geq \ldots \geq \lambda_{np} \geq 0, \end{gathered}$$

$\mu_{jj'} \geq 0$.

# Multiple allocation MF OMP

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where,

$$X = \{x_1, \ldots, x_p\},$$

$$d_i(x) = \|a_i - x\|_\tau,$$

$$d_{(1)}(x) \geq d_{(2)}(x) \geq \ldots \geq d_{(n)}(x),$$

$$\begin{gathered} \lambda_{11} \geq \lambda_{21} \geq \ldots \geq \lambda_{n1} \geq 0, \\ \lambda_{12} \geq \lambda_{22} \geq \ldots \geq \lambda_{n2} \geq 0, \\ \ldots \\ \lambda_{1p} \geq \lambda_{2p} \geq \ldots \geq \lambda_{np} \geq 0, \end{gathered}$$

$\mu_{jj'} \geq 0$.

$$\rho_\lambda^{NI} := \min_x \{ f_\lambda^{NI}(x_1, \ldots, x_p) : \ x_j \in \mathbb{R}^d, \forall j = 1, \ldots, p\}, \qquad (\mathbf{LOCOMF-NI})$$

# Decision variables

$v_{ij}$ and $w_{\ell j}$ continuous variables modelling the ordered weighted average in the o.f.,

$$x_j = \text{coordinates in } \mathbb{R}^d \text{ of the } j\text{-th facility},$$

$$y_{ijk} = \text{ absolute value of the } k\text{-th coordinate of the vector } x_j - a_i,$$

$$u_{ij} = \|x_j - a_i\|_\tau^\tau.$$

$$z_{jj'k} = \text{ absolute value of the } k\text{-th coordinate of the vector } x_j - x_{j'},$$

$$t_{jj'} = \|x_j - x_{j'}\|_\tau^\tau.$$

# Multiple allocation MF OMP

## Theorem

Let $\tau = \frac{r}{s}$ be such that $r, s \in \mathbb{N} \setminus \{0\}$, $r \geq s$ and $\gcd(r, s) = 1$. For any set of lambda weights satisfying $\lambda_{1j} \geq \ldots \geq \lambda_{nj} \geq 0$ for all $j = 1, \ldots, p$, Problem (**LOCOMF − NI**) is equivalent to

$$\rho_{\lambda}^{NI} = \min \quad \sum_{\mathbf{i=1}}^{\mathbf{n}} \sum_{\mathbf{j=1}}^{\mathbf{p}} \mathbf{v_{ij}} + \sum_{\boldsymbol{\ell}\mathbf{=1}}^{\mathbf{n}} \sum_{\mathbf{j=1}}^{\mathbf{p}} \mathbf{w_{\ell j}} + \sum_{j=1}^{p-1} \sum_{j'=j+1}^{p} t_{jj'} \tag{2}$$

$$\begin{array}{lll}
s.t & \mathbf{v_{ij}} + \mathbf{w_{\ell j}} \geq \lambda_{\ell \mathbf{j}} \mathbf{u_{ij}}, & \forall i, \ell = 1, \ldots, n, \ j = 1, \ldots, p \\
& y_{ijk} - x_{jk} + a_{ik} \geq 0, & i = 1, \ldots, n, \ j = 1, \ldots, p, \ k = 1, \ldots, d \\
& y_{ijk} + x_{jk} - a_{ik} \geq 0, & i = 1, \ldots, n, \ j = 1, \ldots, p, \ k = 1, \ldots, d, \\
& \mathbf{y_{ijk}^{r}} \leq \varsigma_{\mathbf{ijk}}^{\mathbf{s}} \mathbf{u_{ij}^{r-s}}, & i = 1, \ldots, n, \ j = 1, \ldots, p, \ k = 1, \ldots, d, \\
& \omega_i^{\frac{r}{s}} \sum_{k=1}^{d} \varsigma_{ijk} \leq u_{ij}, & i = 1, \ldots, n, \ j = 1, \ldots, p \\
& z_{jj'k} - x_{jk} + x_{j'k} \geq 0, & j, j' = 1, \ldots, p, \ k = 1, \ldots, d, \\
& z_{jj'k} + x_{jk} - x_{j'k} \geq 0, & j, j' = 1, \ldots, p, \ k = 1, \ldots, d, \\
& \mathbf{z_{jj'k}^{r}} \leq \xi_{\mathbf{jj'k}}^{\mathbf{s}} \mathbf{t_{jj'}^{r-s}}, & j, j' = 1, \ldots, p, \ k = 1, \ldots, d, \\
& \mu_{jj'}^{\frac{r}{s}} \sum_{k=1}^{d} \xi_{jj'k} \leq t_{jj'}, & j, j' = 1, \ldots, p, \\
& \varsigma_{ijk} \geq 0, & i = 1, \ldots, n, \ j = 1, \ldots, p, \ k = 1, \ldots, d, \\
& \xi_{jj'k} \geq 0 & j, j' = 1, \ldots, p, \ k = 1, \ldots, d.
\end{array}$$

# Multiple allocation MF OMP

## Theorem

Let $\tau = \frac{r}{s}$ be such that $r, s \in \mathbb{N} \setminus \{0\}$, $r \geq s$ and $\gcd(r, s) = 1$. For any set of lambda weights satisfying $\lambda_{1j} \geq \ldots \geq \lambda_{nj} \geq 0$ for all $j = 1, \ldots, p$, Problem (**LOCOMF − NI**) is equivalent to

$$\rho_\lambda^{NI} = \min \quad \sum_{\mathbf{i=1}}^{\mathbf{n}} \sum_{\mathbf{j=1}}^{\mathbf{p}} \mathbf{v_{ij}} + \sum_{\boldsymbol{\ell}\mathbf{=1}}^{\mathbf{n}} \sum_{\mathbf{j=1}}^{\mathbf{p}} \mathbf{w}_{\boldsymbol{\ell}\mathbf{j}} + \sum_{j=1}^{p-1} \sum_{j'=j+1}^{p} t_{jj'} \tag{2}$$

$$\begin{aligned}
s.t \qquad & \mathbf{v_{ij}} + \mathbf{w}_{\boldsymbol{\ell}\mathbf{j}} \geq \boldsymbol{\lambda}_{\boldsymbol{\ell}\mathbf{j}} \mathbf{u_{ij}}, && \forall i, \ell = 1, \ldots, n, \; j = 1, \ldots, p \\
& y_{ijk} - x_{jk} + a_{ik} \geq 0, && i = 1, \ldots, n, \; j = 1, \ldots, p, \; k = 1, \ldots, d \\
& y_{ijk} + x_{jk} - a_{ik} \geq 0, && i = 1, \ldots, n, \; j = 1, \ldots, p, \; k = 1, \ldots, d, \\
& \mathbf{y_{ijk}^r} \leq \varsigma_{\mathbf{ijk}}^{\mathbf{s}} \mathbf{u_{ij}^{r-s}}, && i = 1, \ldots, n, \; j = 1, \ldots, p, \; k = 1, \ldots, d, \\
& \omega_i^{\frac{r}{s}} \sum_{k=1}^{d} \varsigma_{ijk} \leq u_{ij}, && i = 1, \ldots, n, \; j = 1, \ldots, p \\
& z_{jj'k} - x_{jk} + x_{j'k} \geq 0, && j, j' = 1, \ldots, p, \; k = 1, \ldots, d, \\
& z_{jj'k} + x_{jk} - x_{j'k} \geq 0, && j, j' = 1, \ldots, p, \; k = 1, \ldots, d, \\
& \mathbf{z_{jj'k}^r} \leq \xi_{\mathbf{jj'k}}^{\mathbf{s}} \mathbf{t_{jj'}^{r-s}}, && j, j' = 1, \ldots, p, \; k = 1, \ldots, d, \\
& \mu_{jj'}^{\frac{r}{s}} \sum_{k=1}^{d} \xi_{jj'k} \leq t_{jj'}, && j, j' = 1, \ldots, p, \\
& \varsigma_{ijk} \geq 0, && i = 1, \ldots, n, \; j = 1, \ldots, p, \; k = 1, \ldots, d, \\
& \xi_{jj'k} \geq 0 && j, j' = 1, \ldots, p, \; k = 1, \ldots, d.
\end{aligned}$$

# Multiple allocation MF OMP

## Theorem

Let $\tau = \frac{r}{s}$ be such that $r, s \in \mathbb{N} \setminus \{0\}$, $r \geq s$ and $\gcd(r,s) = 1$. For any set of lambda weights satisfying $\lambda_{1j} \geq \ldots \geq \lambda_{nj} \geq 0$ for all $j = 1, \ldots, p$, Problem $(\mathbf{LOCOMF - NI})$ is equivalent to

$$\rho_\lambda^{NI} = \min \quad \sum_{\mathbf{i=1}}^{\mathbf{n}} \sum_{\mathbf{j=1}}^{\mathbf{p}} \mathbf{v_{ij}} + \sum_{\boldsymbol{\ell}=\mathbf{1}}^{\mathbf{n}} \sum_{\mathbf{j=1}}^{\mathbf{p}} \mathbf{w}_{\boldsymbol{\ell}\mathbf{j}} + \sum_{j=1}^{p-1} \sum_{j'=j+1}^{p} t_{jj'} \tag{2}$$

$$\begin{array}{llr}
s.t & \mathbf{v_{ij}} + \mathbf{w}_{\boldsymbol{\ell}\mathbf{j}} \geq \lambda_{\boldsymbol{\ell}\mathbf{j}} \mathbf{u_{ij}}, & \forall i, \ell = 1, ..., n, \ j = 1, \ldots, p \\
& y_{ijk} - x_{jk} + a_{ik} \geq 0, & i = 1, \ldots, n, \ j = 1, ..., p, \ k = 1, \ldots, d \\
& y_{ijk} + x_{jk} - a_{ik} \geq 0, & i = 1, \ldots, n, \ j = 1, ..., p, \ k = 1, \ldots, d, \\
& \mathbf{y_{ijk}^r} \leq \boldsymbol{\varsigma}_{\mathbf{ijk}}^{\mathbf{s}} \mathbf{u_{ij}^{r-s}}, & i = 1, \ldots, n, \ j = 1, ..., p, \ k = 1, \ldots, d, \\
& \omega_i^{\frac{r}{s}} \sum_{k=1}^{d} \varsigma_{ijk} \leq u_{ij}, & i = 1, \ldots, n, \ j = 1, \ldots, p \\
& z_{jj'k} - x_{jk} + x_{j'k} \geq 0, & j, j' = 1, \ldots, p, \ k = 1, ..., d, \\
& z_{jj'k} + x_{jk} - x_{j'k} \geq 0, & j, j' = 1, \ldots, p, \ k = 1, ..., d, \\
& \mathbf{z_{jj'k}^r} \leq \boldsymbol{\xi}_{\mathbf{jj'k}}^{\mathbf{s}} \mathbf{t_{jj'}^{r-s}}, & j, j' = 1, \ldots, p, \ k = 1, \ldots, d, \\
& \mu_{jj'}^{\frac{r}{s}} \sum_{k=1}^{d} \xi_{jj'k} \leq t_{jj'}, & j, j' = 1, \ldots, p, \\
& \varsigma_{ijk} \geq 0, & i = 1, \ldots, n, \ j = 1, \ldots, p, \ k = 1, \ldots, d, \\
& \xi_{jj'k} \geq 0 & j, j' = 1, \ldots, p, \ k = 1, \ldots, d.
\end{array}$$

# Sorting with permutations: $\lambda_1 \geq \ldots \geq \lambda_n$

$$\sum_{i=1}^{n}\sum_{j=1}^{p}\lambda_{ij}u_{(i)j} = \max_{\sigma\in\mathcal{P}_n}\sum_{i=1}^{n}\sum_{j=1}^{p}\lambda_{ij}u_{\sigma(i)j}.$$

The permutations in $\mathcal{P}_n$ can be represented by the binary variables $s_{ijk} \in \{0,1\}$
Next, combining the two sets of variables the objective function is:

$$\left\{\begin{array}{rl} \sum_{i=1}^{n}\sum_{j=1}^{p}\lambda_{ij}u_{(i)j} = & \max \sum_{i=1}^{n}\sum_{j=1}^{p}\sum_{k=1}^{n}\lambda_{ij}u_{ij}s_{ijk} \\ & s.t \quad \sum_{i=1}^{n} s_{ijk} = 1,\ \forall j=1,\ldots,p,\ k=1,\ldots,n, \\ & \sum_{k=1}^{n} s_{ijk} = 1,\ \forall i=1,\ldots,n,\ j=1,\ldots,p, \\ & s_{ijk}\in\{0,1\}. \end{array}\right. \tag{3}$$

# Sorting with a linear program

For fixed $j \in \{1, \ldots, p\}$ and for any vector $u_{\cdot j} \in \mathbb{R}^n$, by using the dual of the assignment problem (7) we obtain the following expression

$$\left\{ \begin{array}{ll} \sum_{i=1}^{n} \lambda_{ij} u_{(i)j} = & \min \sum_{i=1}^{n} v_{ij} + \sum_{l=1}^{n} w_{lj} \\ & s.t \quad v_{ij} + w_{lj} \geq \lambda_{lj} u_{ij}, \ \forall i, l = 1, \ldots, n. \end{array} \right. \tag{4}$$

Finally, we replace (4) for the objective function and we get

$$\left\{ \begin{array}{ll} \min \sum_{i=1}^{n} \sum_{j=1}^{p} v_{ij} + \sum_{l=1}^{n} \sum_{j=1}^{p} w_{lj} + \sum_{j=1}^{p-1} \sum_{j'=j+1}^{p} t_{jj'} & \\ s.t \quad v_{ij} + w_{lj} \geq \lambda_{lj} u_{ij}, & \forall i, l = 1, \ldots, n, \ j = 1, \ldots, p, \\ \qquad u_{ij} \geq \omega_i \|x_j - a_i\|_\tau, & i = 1, \ldots, n, \ j = 1, \ldots, p, \\ \qquad t_{jj'} \geq \mu_{jj'} \|x_j - x_{j'}\|_\tau, & j, j' = 1, \ldots, p. \end{array} \right. \tag{5}$$

# Second Order Cone Constraints

## Lemma (Blanco, P., ElHaj-BenAli, 2014)

Let $\tau = \frac{r}{s}$ be such that $r, s \in \mathbb{N} \setminus \{0\}$, $r > s$, $\gcd(r,s) = 1$ and $k = \lceil \log_2(r) - 1 \rceil$.
Let $x$, $u$ and $t$ be non negative and satisfying

$$x^r \leq u^s t^{r-s}. \tag{6}$$

Then, there exists $w$ such that, (6) is equivalent to:

$$\begin{cases} w_1^2 & \leq A_1 B_1, \\ \vdots & \vdots \\ w_m^2 & \leq A_m B_m, \\ x^2 & \leq A_{m+1} B_{m+1}, \end{cases} \tag{7}$$

where $A_i, B_i \in \{w_{i-1}, u, t, x\}$ for $i = 1, \ldots, m$ and
$m = 1 + 2\#\{i : \alpha_i + \beta_i + \gamma_i \geq 2, 1 \leq i < k-1\} + \#\{i : \alpha_i + \beta_i + \gamma_i \leq 1, 1 \leq i < k-1\} \sim \mathcal{O}(log(r))$ with:

$$\begin{aligned} s &= \alpha_{k-1}2^{k-1} + \alpha_{k-2}2^{k-2} + \ldots + \alpha_1 2^1 + \alpha_0 2^0, \\ r - s &= \beta_{k-1}2^{k-1} + \beta_{k-2}2^{k-2} + \ldots + \beta_1 2^1 + \beta_0 2^0, \\ 2^k - r &= \gamma_{k-1}2^{k-1} + \gamma_{k-2}2^{k-2} + \ldots + \gamma_1 2^1 + \gamma_0 2^0, \\ 2^k &= (\alpha_{k-1} + \beta_{k-1} + \gamma_{k-1})2^{k-1} + \cdots + (\alpha_0 + \beta_0 + \gamma_0)2^0, \end{aligned}$$

$\alpha_i,\ \beta_i,\ \gamma_i \in \{0, 1\}$.

## Corollary (sequel)

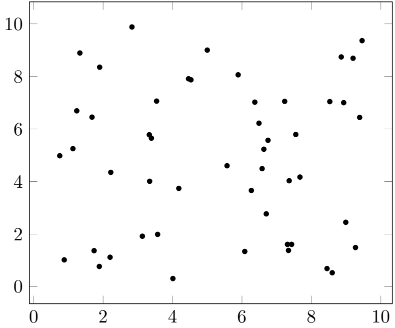
Moreover, Problem (2) satisfies Slater condition and it can be represented as a second order cone (or semidefinite) program with $(np + p^2)(2d + 1) + p^2$ linear inequalities and at most $4(p^2 d + npd) \log r$ second order cone (or linear matrix) inequalities.

## Remark

As consequence of the previous theorem, problem (2) is convex and can be solved, up to any given accuracy, in polynomial time for any dimension $d$.

# Experiments

Coded in: Gurobi 5.6.
Intel Core i7 processor; 2x 2.40 GHz; 4 GB of RAM
Instance: 50-points data set in Eilon, Watson-Gandy & Christofides (1971).



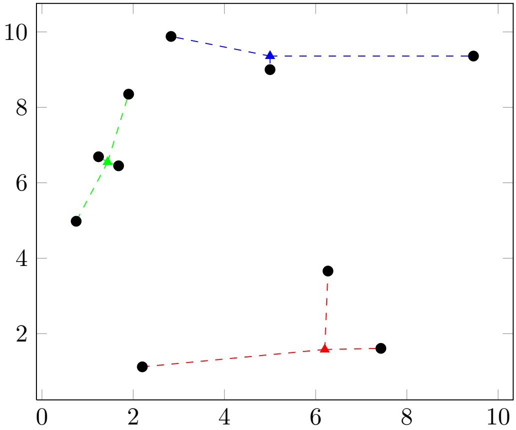

CPU-Times (secs.):

| $p$ \ $\tau$ | 1.5 | 2 | 3 |
|---|---|---|---|
| 2 | 2.51 | 2.12 | 3.75 |
| 5 | 12.78 | 6.52 | 9.81 |
| 10 | 29.19 | 10.57 | 19.55 |
| 15 | 49.49 | 19.11 | 40.45 |
| 30 | 148.75 | 40.56 | 85.57 |

# The program continues with some extensions...

# A hard MF problem: Single allocation MF OMP

Miehle (1958), Cooper (1963)
C. Pey-Chun, P. Hansen, B. Jaumard and T. Hoang (1998).



# Single allocation MF OMP

$$\begin{array}{rlcl} \tilde{f}_i(x): & \mathbb{R}^{pd} & \mapsto & \mathbb{R} \\ & x = (x_1, \ldots, x_p) & \mapsto & t_i := \min_{j=1..p} \{\|x_j - a_i\|_\tau\}. \end{array}$$

Assume that:

- $\mathbf{K} \subseteq \mathbb{R}^d$.
- $\tau := \frac{r}{s} \geq 1,\ r, s \in \mathbb{N}$ with $gcd(r, s) = 1$.
- $\lambda_\ell \geq 0$ for all $\ell = 1, \ldots, n$.

Consider the following problem

$$\rho_\lambda := \min_x \{\sum_{i=1}^{n} \lambda_i \tilde{f}_{(i)}(x) : x = (x_1, \ldots, x_p),\ x_j \in \mathbf{K},\ \forall j = 1, \ldots, p\}, \quad (\mathbf{LOCOMF})$$

# Single allocation MF OMP

$$\hat{\rho}_\lambda = \min \sum_{\ell=1}^{n} \lambda_\ell \theta_\ell \qquad (\mathrm{MFOMP}_\lambda)$$

$$\text{s.t. } h^1_{il} := t_i \le \theta_\ell + UB_i(1 - w_{i\ell}), \qquad i = 1, \ldots, n,\ \ell = 1, \ldots, n,$$

$$h^2_l := \theta_\ell \ge \theta_{\ell+1}, \qquad \ell = 1, \ldots, n-1,$$

$$h^3_{ij} := u_{ij} \le t_i + UB_i(1 - z_{ij}), \qquad \forall\ i = 1, \ldots, n,\ j = 1, \ldots, p,$$

$$h^4_{ijk} := v_{ijk} - x_{jk} + a_{ik} \ge 0, \qquad i = 1, \ldots, n,\ j = 1, \ldots, p,\ k = 1, \ldots, d,$$

$$h^5_{ijk} := v_{ijk} + x_{jk} - a_{ik} \ge 0, \qquad i = 1, \ldots, n,\ j = 1, \ldots, p,\ k = 1, .., d,$$

$$h^6_{ijk} := v^r_{ijk} \le \zeta^s_{ijk} u^{r-s}_{ij}, \qquad i = 1, \ldots, n,\ j = 1, \ldots, p,\ k = 1, \ldots, d,$$

$$h^7_{ij} := \sum_{k=1}^{d} \zeta_{ijk} \le u_{ij}, \qquad i = 1, \ldots, n,\ j = 1, \ldots, p,$$

$$h^8_i := \sum_{j=1}^{p} z_{ij} = 1, \qquad i = 1, \ldots, n,$$

$$h^9_l := \sum_{i=1}^{n} w_{il} = 1, \qquad l = 1, \ldots, n,$$

$$h^{10}_i := \sum_{l=1}^{n} w_{il} = 1, \qquad i = 1, \ldots, n,$$

$$w_{i\ell} \in \{0, 1\}, \qquad \forall\ i, \ell = 1, \ldots, n,$$

$$z_{ij} \in \{0, 1\}, \qquad \forall\ i = 1, \ldots, n, j = 1, \ldots, p,$$

$$\theta_\ell,\ t_i,\ v_{ijk},\ \zeta_{ijk},\ u_{ij} \in \mathbb{R}^+, \qquad i, l = 1, \ldots, n, j = 1, \ldots, p, k = 1, \ldots, d,$$

$$x_j \in \mathbf{K}, \qquad j = 1, \ldots, p.$$

## Theorem

Let $x$ be a feasible solution of **LOCOMF** then there exists a solution $(x, z, u, v, \zeta, w, t, \theta)$ for $\mathrm{MFOMP}_\lambda$ such that their objective values are equal. Conversely, if $(x, z, u, v, \zeta, w, t, \theta)$ is a feasible solution for $\mathrm{MFOMP}_\lambda$ then $x$ is a feasible solution for **LOCOMF**. Furthermore, if $K$ satisfies Slater condition then the feasible region of the continuous relaxation of $\mathrm{MFOMP}_\lambda$ also satisfies Slater condition and $\rho_\lambda = \hat{\rho}_\lambda$.

# Experiments

| Problem | $p$ | $\tau$ | CPU Time | $f^*$ | Problem | $p$ | $\tau$ | CPU Time | $f^*$ | Problem | $p$ | $\tau$ | CPU Time | $f^*$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $p$-median | 2 | 1.5 | 22.31 | 150.955 | $p$-center | 2 | 1.5 | 1.03 | 4.9452 | $p$-25-centrum | 2 | 1.5 | 10.08 | 100.8474 |
| | | 2 | 1.13 | 135.5222 | | | 2 | 0.28 | 4.8209 | | | 2 | 0.38 | 95.0892 |
| | | 3 | 23.68 | 130.856 | | | 3 | 13.51 | 4.788 | | | 3 | 139.03 | 89.0238 |
| | 5 | 1.5 | 55.28 | 78.6074 | | 5 | 1.5 | 3.73 | 2.8831 | | 5 | 1.5 | 33.09 | 53.4995 |
| | | 2 | 12.49 | 72.2369 | | | 2 | 5.37 | 2.661 | | | 2 | 7.61 | 49.6932 |
| | | 3 | 125.1 | 68.1791 | | | 3 | 2.87 | 2.5094 | | | 3 | 18.23 | 46.9844 |
| | 10 | 1.5 | 5.36 | 45.0525 | | 10 | 1.5 | 2.66 | 1.6929 | | 10 | 1.5 | 68.36 | 30.7137 |
| | | 2 | 2.31 | 41.6851 | | | 2 | 5.3 | 1.6113 | | | 2 | 17.93 | 28.9017 |
| | | 3 | 4.76 | 39.7222 | | | 3 | 55.76 | 1.595 | | | 3 | 225.64 | 27.5376 |
| | 15 | 1.5 | 6.7 | 30.0543 | | 15 | 1.5 | 9.44 | 1.1139 | | 15 | 1.5 | 49.92 | 22.4165 |
| | | 2 | 43.91 | 27.6282 | | | 2 | 0.62 | 1.0717 | | | 2 | 11.26 | 20.6536 |
| | | 3 | 150.99 | 26.6047 | | | 3 | 50.08 | 1.053 | | | 3 | 244.59 | 20.8544 |
| | 30 | 1.5 | 14.45 | 9.9488 | | 30 | 1.5 | 74.43 | 1.008 | | 30 | 1.5 | 202.54 | 9.0806 |
| | | 2 | 4.81 | 8.7963 | | | 2 | 1.53 | 0.9192 | | | 2 | 5.29 | 8.521662 |
| | | 3 | 198.78 | 8.6995 | | | 3 | 57.37 | 0.8508 | | | 3 | 287.90 | 8.001695 |

Table: Computational results for $p$-median, $p$-center and $p$-25-centrum problems for the 50-points in Eilon, Watson-Gandy & Christofides data set.

# Constrained MF Ordered Median Problem

A SDP relaxation of the LOCOMF

Let $h_0(\theta) := \sum_{\ell=1}^{m} \lambda_\ell \theta_\ell$, and denote $\xi_j := \lceil(\deg g_j)/2\rceil$ and $\nu_j := \lceil(\deg h_j)/2\rceil$, where $\{g_1, \ldots, g_{n_{\mathbf{K}}}\}$, and $\{h_1, \ldots, h_{nc1}\}$ are, respectively, the polynomial constraints that define $\mathbf{K}$ and $\hat{\mathbf{K}} \setminus \mathbf{K}$ in $\mathrm{MFOMP}_\lambda$. For $r \geq r_0 := \max\{\max_{k=1,\ldots,n_{\mathbf{K}}} \xi_k, \max_{j=0,\ldots,nc1} \nu_j\}$, introduce the hierarchy of semidefinite programs:

$$
\begin{array}{lllr}
\min_{\mathbf{y}} & \mathrm{L}_{\mathbf{y}}(p_\lambda) & & \\
\text{s.t.} & \mathrm{M}_r(\mathbf{y}) & \succeq 0, & \\
 & \mathrm{M}_{r-\xi_k}(g_k, \mathbf{y}) & \succeq 0, \quad k = 1, \ldots, n_{\mathbf{K}}, & (\mathbf{Q1}_r) \\
 & \mathrm{M}_{r-\nu_j}(h_j, \mathbf{y}) & \succeq 0, \quad j = 1, \ldots, nc1, & \\
 & y_0 = 1, & &
\end{array}
$$

## Theorem

Let $\hat{\mathbf{K}} \subset \mathbb{R}^{nv_1}$(compact) be the feasible domain of Problem $\mathrm{MFOMP}_\lambda$. Let inf $\mathbf{Q1}_r$ be the optimal value of the semidefinite program $\mathbf{Q1}_r$. Then, with the notation above:

(a) inf $\mathbf{Q1}_r \uparrow \rho_\lambda$ as $r \to \infty$.

(b) Let $\mathbf{y}^r$ be an optimal solution of the SDP relaxation $\mathbf{Q1}_r$. If

$$\operatorname{rank} \mathrm{M}_r(\mathbf{y}^r) = \operatorname{rank} \mathrm{M}_{r-r_0}(\mathbf{y}^r) = \varphi$$

then min $\mathbf{Q1}_r = \rho_\lambda$ and one may extract $\varphi$ points $(x_1^*(k), \ldots, x_p^*(k), z^*(k), u^*(k), v^*(k), \zeta^*(k), w^*(k), t^*(k), \theta^*(k))_{k=1}^{\varphi} \subset \hat{\mathbf{K}}$, all global minimizers of the $\mathrm{MFOMP}_\lambda$ problem.

- The size of SDP relaxations grows rapidly with the original problem size.
- In view of the present status of SDP-solvers, only small to medium size problems can be solved.

We found in our model patterns to exploit:

- Sparsity.
- Symmetry.

Thank you!

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http://arxiv.org/abs/1401.0817