Disjunctive Conic and Cylindrical Cuts in Solving Conic Integer Portfolio Optimization Problems

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Outline

Mixed Integer Second Order Cone Optimization

$$\begin{aligned} \min \quad & c^T x \\ \text{s.t.} \quad & Ax = b \qquad\qquad \text{(MISOCO)} \\ & x \in \mathbb{K} \\ & x \in \mathbb{Z}^d \times \mathbb{R}^{n-d}, \end{aligned}$$

where,

- $A \in \mathbb{R}^{m \times n}$, $c \in \mathbb{R}^n$, $b \in \mathbb{R}^m$
- $\mathbb{K} = \mathbb{L}^{n_1} \oplus \ldots \oplus \mathbb{L}^{n_k}$
- $\mathbb{L}^{n_i} = \{x \mid x_1 \geq \|x_{2:n_i}\|\}$
- Rows of A are linearly independent

Algorithmic Framework

Branch and Conic Cut (B&CC) Algorithm

- Similar to a standard branch-and-cut algorithm.
  - Solve the continuous relaxation (a SOCO problem).
  - Identify a violated disjunction (fractional variable).
  - Either branch or generate a disjunctive constraint.
- The convex hull of the disjunctive set associated with a variable disjunction in MISOCO can be obtained by adding a single conic constraint, called **Disjunctive Conic Cut (DCC)**.
- Procedure for cut generation is similar to lift and project for mixed integer linear optimization (MILO) problems, **but** we generate novel Disjunctive Conic Cuts **(DCCs)**.
- A DCC can be computed efficiently.

Step 1: Solve the relaxed problem

Find the optimal solution x^*_{SOCO} for the continuous relaxation

$$\begin{array}{rrrrrl} \text{min:} & 3x_1 & +2x_2 & +2x_3 & +x_4 & \\ \text{s.t.:} & 9x_1 & +x_2 & +x_3 & +x_4 & = 10 \\ & & (x_1, x_2, x_3, x_4) & & \in \mathbb{L}^4 & \\ & & & x_4 & \in \mathbb{Z}. & \end{array}$$

Relaxing the integrality constraint we get the optimal solution:

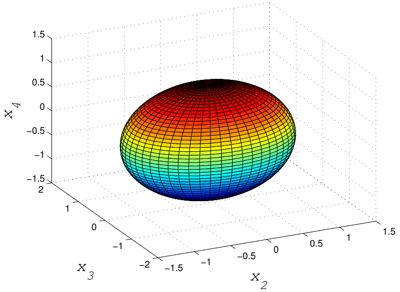
$$x^*_{\text{SOCO}} = (1.36, -0.91, -0.91, -0.45),$$

with and optimal objective value: $z^* = 0.00$.

Reformulation

Reformulation of the relaxed problem

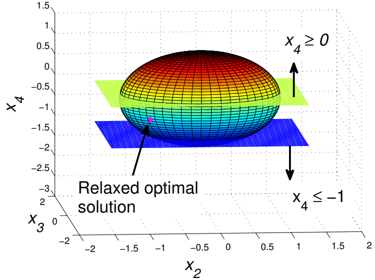
$$
\begin{aligned}
\text{min:} \quad & \frac{1}{3}\left(10 + 5x_2 + 5x_3 + 2x_4\right) \\
\text{s.t.:} \quad & \begin{bmatrix} x_2 & x_3 & x_4 \end{bmatrix} \begin{bmatrix} 8 & -\frac{1}{10} & -\frac{1}{10} \\ -\frac{1}{10} & 8 & -\frac{1}{10} \\ -\frac{1}{10} & -\frac{1}{10} & 8 \end{bmatrix} \begin{bmatrix} x_2 \\ x_3 \\ x_4 \end{bmatrix} + 2\begin{bmatrix} 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} x_2 \\ x_3 \\ x_4 \end{bmatrix} - 10 \leq 0 \\
& x_4 \in \mathbb{Z} .
\end{aligned}
$$



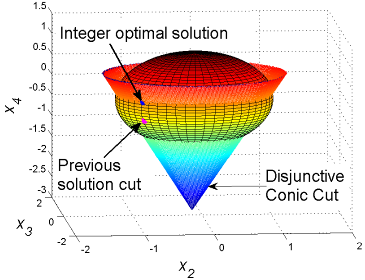
Feasible set of the reformulated problem

Step 2: Find a violated disjunction

The disjunction $x_4 \le -1 \ \bigvee \ x_4 \ge 0$ is violated by x^*_{soco}



(A) Disjunction



(B) Disjunctive conic cut

An integer optimal solution is obtained after adding one cut:

$$x^*_{\text{misoco}} = x^*_{\text{soco}} = (1.32, \ -0.93, \ -0.93, \ 0.00, \ 10.06, \ -10.06, \ 0.00),$$

with an optimal objective value: $z^*_{\text{misoco}} = x^*_{\text{soco}} = 0.24.$

Related literature

Related literature about conic cuts for MISOCO problems:

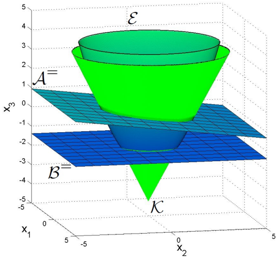
- Belotti et al. (2011, 2013), A conic representation of the convex hull of disjunctive sets and conic cuts for integer second order cone optimization.
- Atamtürk and Narayanan (2010), Conic mixed-integer rounding cuts.
- Dadush, Dey and Vielma (2011), Split closure of an ellipsoid.
- Modaresi, Kılınç and Vielma (2013), Intersection cuts for nonlinear integer programming.
- Modaresi, Kılınç and Vielma (2013), Split cuts for mixed integer conic quadratic programming.
- Kilinc-Karzan (2014), On Minimal Valid Inequalities for Mixed Integer Conic Programs.

Disjunctive conic cuts

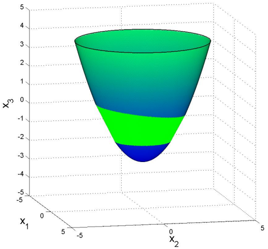
Definition

A closed convex cone $\mathcal{K} \in \mathbb{R}^n$ with $\dim(\mathcal{K}) > 1$ is called a *Disjunctive Conic Cut* (DCC) for $\mathcal{E}$ and the disjunctive set $\mathcal{A} \cup \mathcal{B}$ if

$$\operatorname{conv}(\mathcal{E} \cap (\mathcal{A} \cup \mathcal{B})) = \mathcal{E} \cap \mathcal{K}.$$



(A)



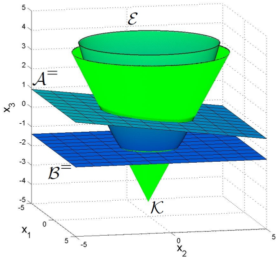
(B)

Disjunctive conic cuts

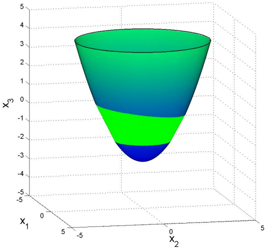
Proposition

A closed convex cone $\mathcal{K} \in \mathbb{R}^n$ *with* $\dim(\mathcal{K}) > 1$ *is a (unique) DCC for* $\mathcal{E}$ *and the disjunctive set* $\mathcal{A} \cup \mathcal{B}$*, if*

$$\mathcal{K} \cap \mathcal{A}^= = \mathcal{E} \cap \mathcal{A}^= \quad \text{and} \quad \mathcal{K} \cap \mathcal{B}^= = \mathcal{E} \cap \mathcal{B}^=.$$



(A)



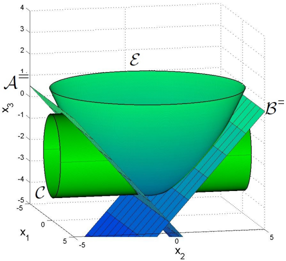
(B)

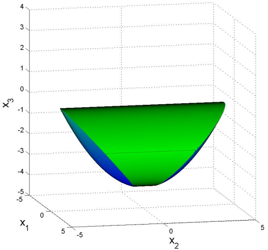
Disjunctive cylindrical cuts

Definition

Let $\mathcal{E}$ be a closed convex set. A closed convex cylinder $\mathcal{C}$ is a *Disjunctive Cylindrical Cut* (DCyC) for the set $\mathcal{E}$ and the disjunctive set $\mathcal{A} \cup \mathcal{B}$ if

$$\operatorname{conv}(\mathcal{E} \cap (\mathcal{A} \cup \mathcal{B})) = \mathcal{E} \cap \mathcal{C}.$$



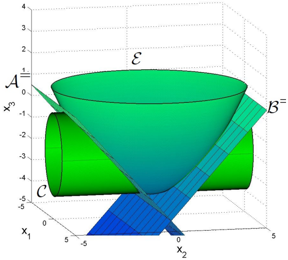


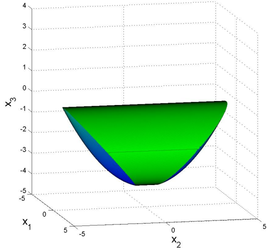
Disjunctive cylindrical cuts

Proposition

A convex cylinder $\mathcal{C} \in \mathbb{R}^n$ with a unique direction $d^0 \in \mathbb{R}^n$, such that $a^\top d^0 \neq 0$ and $b^\top d^0 \neq 0$, is a (unique) DCyC for $\mathcal{E}$ and the disjunctive set $\mathcal{A} \cup \mathcal{B}$, if

$$\mathcal{C} \cap \mathcal{A}^= = \mathcal{E} \cap \mathcal{A}^= \quad \text{and} \quad \mathcal{C} \cap \mathcal{B}^= = \mathcal{E} \cap \mathcal{B}^=.$$



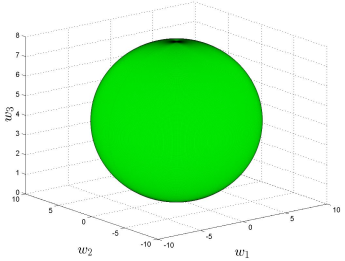


Quadrics

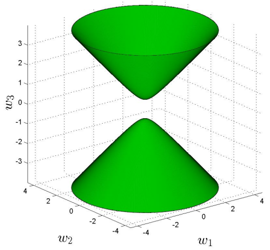
Definition

Let $P \in \mathbb{R}^{\ell \times \ell}$, $p, w \in \mathbb{R}^{\ell}$ and $\rho \in \mathbb{R}$, then the quadric $\mathcal{Q}$ is the set defined as

$$\mathcal{Q} = \{w \in \mathbb{R}^{\ell} \mid w^\top P w + 2p^\top w + \rho \leq 0\}.$$



Ellipsoid.



Hyperboloid.

Uni-parametric family of quadrics

Theorem

Let (P, p, ρ) be a quadric and consider two hyperplanes

$$\mathcal{A}^{=} = \{z \mid a^{\top} z = \alpha\} \text{ and } \mathcal{B}^{=} = \{z \mid d^{\top} z = \beta\}.$$

The family of quadrics $(P(\tau), p(\tau), \rho(\tau))$ parametrized by $\tau \in \mathbb{R}$ having the same intersection with $\mathcal{A}^{=}$ and $\mathcal{B}^{=}$ as the quadric (P, p, ρ) is given by

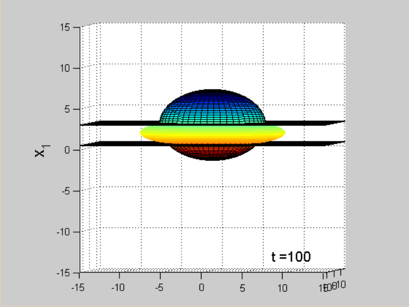
$$P(\tau) = P + \tau \frac{ad^T + da^T}{2}$$

$$p(\tau) = p - \tau \frac{\beta a + \alpha d}{2}$$

$$\rho(\tau) = \rho + \tau \alpha \beta.$$

Family of quadrics with parallel hyperplanes

$$\mathcal{Q}(\tau) = \{w \in \mathbb{R}^n \mid w^\top P(\tau) w + 2p(\tau)^\top w + \rho(\tau) \leq 0\}$$



Sequence of quadrics for $-101 \leq \tau \leq 100$.

For the classification of $\mathcal{Q}(\tau)$ we use a criteria based on:

- the value of

$$p(\tau)^\top P(\tau)^{-1} p(\tau) - \rho(\tau) = \frac{f(\tau)}{h(\tau)},$$

where $f(\tau)$ is a second degree polynomial in τ, $h(\tau)$ is a linear function of τ;

- the roots $\bar{\tau}_1 \leq \bar{\tau}_2$ of $f(\tau)$, and the root $\hat{\tau}$ of $h(\tau)$.

Computational experience

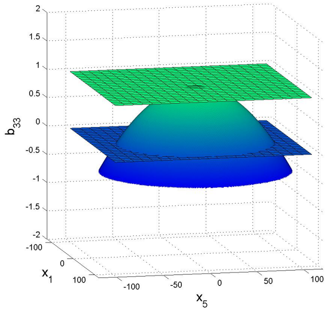
- Constrained layout problems, CLay Problems (Bonami et al. 2008).
- Quadratic constraints corresponding to Euclidean-distance

$$(x_1 - 17.5)^2 + (x_5 - 7)^2 + 6814 * b_{33} \leq 6850.$$

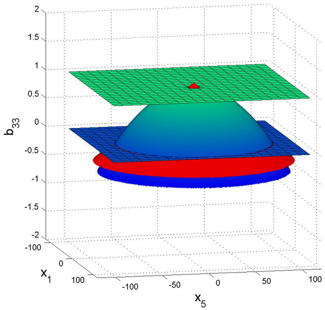
- All integer variables are binary, for example in the illustrative constraint the binary variable is b_{33}.

	0203M	0204M	0205M	0303M	0304M	0305M
Var	31	52	81	34	57	86
Binary	18	21	50	21	36	55
Constraints	55	91	136	67	107	156
Quad	24	32	40	36	48	60

CLay Problems (Bonami et al. 2008)



CLay Quadratic Constraints



DCC cut

This can be done in the preprocessing phase

CLay Problems

Gains when solved with CPLEX 12.4

	0203M	0204M	0205M	0303M	0304M	0305M
Time	48%	63%	23%	53%	41%	55%
Nodes	1%	11%	24%	-6%	47%	20%
Iter	23%	8%	21%	51%	52%	4%

Gains when solved with MOSEK 6.0

	0203M	0204M	0205M	0303M	0304M	0305M
Time	25%	11%	39%	18%	25%	22%
Nodes	17%	-11%	19%	3%	17%	14%
Iter	24%	3%	28%	14%	20%	20%

Classes of Portfolio Optimization

Following classes of Portfolio Optimization leads MISOCO:

1. Cardinality Constraint
2. Logical Constraints
3. Round Lot (Minimum Transaction Lot) Constraint

Classical Mean Variance Portfolio Optimization Problem

- Mean variance portfolio optimization is one of the major models in finance
- x_i denotes the proportion invested in the risky asset i

$$e^\top x = 1 \qquad x \geq 0$$

- Q : variance-covariance matrix for n risky assets
- The objective is minimizing total risk by choosing a portfolio

$$x^\top Q x$$

- R : Prescribed return level
- μ_i : Expected return level of risky asset i
- The portfolio should deliver an expected level of return

$$\mu^\top x \geq R$$

Cardinality Constraint

- Number of assets to be invested can be limited
- K: Number of assets to be invested

$$e^\top z = K$$

- Long-only portfolio

$$0 \leq x \leq z$$

- Long/short portfolio

$$-z \leq x \leq z$$

- Alternative way to define cardinality for long-only portfolios

$$x = \frac{1}{K} z + y$$

$$z \in \{0,1\}^n, y \in \left[0, 1 - \frac{1}{K}\right]$$

Challenges

- Replacing x by $\frac{1}{K}z + y$ leads to PSD matrix $\tilde{Q} \in \mathbb{R}^{2n \times 2n}$

$$x^T Q x \equiv \begin{bmatrix} y \\ z \end{bmatrix}^T \tilde{Q} \begin{bmatrix} y \\ z \end{bmatrix} \leq t$$

  which has rank n.
- Continuous variables (y) makes it difficult to write a cut on this constraint
- We could not find a valid DCC or DCyC for cardinality constrained portfolio optimization problem, yet

Logical Constraints

- There are several types of logical constraints in portfolio optimization problems that lead to binary variables.
- Some of these constraints leads second order conic constraints
- These constraints are used to restrict sector-type investments
- Let $z \in \{0,1\}^n$ be a binary vector that represents investment to the assets. For long-only portfolios, "OR" type of constraint can be represented as

$$z_i z_j \leq 0 \qquad \text{(nonconvex)}$$

- For long/short portfolios let $z \in \{-1,1\}$ and $z \in \mathbb{Z}^N$. Constraint

$$z_i z_j \geq 0$$

  represents asset i and asset j has the same direction. **SOCO representable** – rotated second order cone!
- Future work is to extend to include logical constraints.

Round Lot Constraint (Bonami, Lejeune - 2009)

- In institutional market investors buy lots of assets
- $z_i \in \mathbb{N}^n$: Number of lots bought
- These type of purchases are less risky for investors
- Easy to buy and sell and increase liquidity
- p_i : Proportion of investment of lot to the capital
- Discrete jumps for the 'proportion of the investment' x
- Relation between x and z

$$x_i = p_i z_i$$

- Risk-free investment in money market (x_0) remains continuous
- Money market investment is bounded

$$x_0 < M$$

Mean variance portfolio optimization model

$$
\begin{array}{lrcll}
\text{minimize:} & x^\top Q x & & & \\
\text{subject to:} & \mu_0 x_0 + \mu^\top x & \geq & R & \\
 & x_0 + e^\top x & = & 1 & \\
 & x_0 & \leq & M & \\
 & x_i & = & p_i z_i & \forall i \\
 & x & \geq & 0 & \\
 & z & \in & \mathbb{N}^n, &
\end{array}
$$

- Denote $P = \mathrm{diag}(p)$ and $\bar{Q} = P^\top Q P$.
- Denote $\bar{p} = \sum_i (\mu_i - \mu_0) p_i$ and $\bar{p}_0 = R - \mu_0 \geq 0$.
- Let $\bar{M} = 1 - M$.

- Move the quadratic objective function to the constraint and define new variable t

Revised RL-MVPO

$$
\begin{array}{lrcl}
\text{minimize:} & t & & \\
\text{subject to:} & z^\top \bar{Q} z & \leq & t \\
& \bar{p}^\top z & \geq & \bar{p}_0 \\
& p^\top z & \leq & 1 \\
& p^\top z & \geq & \bar{M} \\
& z & \in & \mathbb{N}^n.
\end{array} \qquad \text{(RL-MVPO)}
$$

Disjunctive Cylindrical Cut Generation

- Disjunctive cylindrical cuts are written on quadratic constraint

$$z^\top \bar{Q} z \leq t$$

- Let z^* be the optimal solution of the continuous relaxation.
- Disjunction on asset j gives us parallel disjunctions

$$A^= = \left\{ \begin{bmatrix} t \\ z \end{bmatrix} \in \mathbb{R}^{N+1} \mid c^\top \begin{bmatrix} t \\ z \end{bmatrix} = \lfloor z_j^* \rfloor \right\}$$

$$B^= = \left\{ \begin{bmatrix} t \\ z \end{bmatrix} \in \mathbb{R}^{N+1} \mid c^\top \begin{bmatrix} t \\ z \end{bmatrix} = \lceil z_j^* \rceil \right\}$$

- Eigenvalue decomposition of $\hat{Q}$ gives

$$\hat{Q} = V^\top \bar{D}^{1/2} J \bar{D}^{1/2} V$$

Disjunctive Cylindrical Cut Generation

- Denote

$$P(\tau) = J + \tau \frac{c}{\|c\|} \left(\frac{c}{\|c\|} \right)^\top$$
$$p(\tau) = \bar{p} - \tau \frac{\alpha + \beta}{2} \frac{c}{\|c\|}$$
$$\rho(\tau) = \tau\alpha\beta,$$

- Intersection of parallel disjunctions $(A^=, B^=)$ and quadratic constraint provide us

$$u^\top P(-1)u + 2p(-1)^\top u + \rho(-1) \leq 0,$$

the disjunctive cylindrical cut, where

$$u = \bar{D}^{1/2} V^\top \begin{bmatrix} t \\ z \end{bmatrix}$$

Preliminary Numerical Experiments

- These cuts are applied to solve a real life instance
- We consider asset allocation problem of 7 countries (Germany, France, Japan, UK, USA, Canada, Australia)
- Equities, bonds and currencies are invested
- Historical data of from January 1971 to August 1991
- US Dollar is used as the reference
- In total there are 20 assets to invest

Preliminary Numerical Experiments

- All disjunctive cuts are added to root node.

Strategies

- Solve continuous relaxations once and add up to 5 cuts

- Cuts are written for assets closest to half
- Results are compared with benchmark case, no cuts added

Contribution to the Running Time and Tree Size Reduction

- $C = 100000,\ R = 0.04$

			Active Cuts				
		Direct	1	1,2	1,2,3	1,2,3,4	1,2,3,4,5
Time	Cont. Rel.	-	0.0188	0.0232	0.0205	0.0214	0.0242
	Cut Generation	-	0	0.073906	0.093006	0.220909	0.343229
	B&B	2.15436	2.06113	2.99248	2.39727	3.43535	2.43902
	Total	2.15436	2.12959	3.13514	2.55068	3.72545	2.85354
Node		7419	5234	6383	3889	5060	3102
Time per Node		2.9E-04	3.9E-04	4.6E-04	6.1E-04	6.7E-04	7.8E-04

Conclusions

- Comprehensive theory of DCC's for MISOCO
- Interesting MISOCO problems arise in many application areas, including location, engineering design, and portfolio management.
- Not all of them are suitable for the DCC methodology
- Encouraging preliminary results for CLAY, random, and Round Lot Constrained portfolio optimization problems
- Cut management, optimizing for solution time is a challenge
- Numerous MISOCO problems to explore in healthcare, location theory, engineering design, robust MILO, ...

Questions?

Thanks for your attention