Robert Weismantel

MINLPs with few integer variables

ETH Zürich

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What and why?

MINLP model

Let $P \subseteq \mathbf{R}^{n+d}$ be a polytope and $f : \mathbf{R}^{n+d} \to \mathbf{R}$ a nonlinear function.

$$\begin{aligned} \min \quad & f(x, y) \\ \text{s.t.} \quad & \\ & (x, y) \in P, \\ & x \in \mathbf{Z}^d, \; y \in \mathbf{R}^n. \end{aligned}$$

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Why study this model?

- (MILP) and (CO) are about to become a technology.
- understand specific class of MINLPs: optimization over continous relaxations is "tractable".
- build a bridge to other areas of mathematics.

The central question

Can we extend theory and algorithms from MILP and NLO to the mixed integer setting?

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What do we aim at?

- Complexity results.
- Algorithmic schemes amenable to an analysis.

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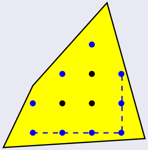
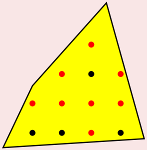
Aspects of nonlinear discrete optimization

Convex maximization



Polynomial optimization



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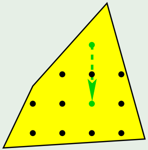

Parametric non-linear optimization and W-mappings

A borderline case from the point of view of computational complexity

$$\begin{array}{ll} \min & f(Wx) \\ \text{s.t.} & x \in P \cap \mathbf{Z}^n \end{array}$$

with $W \in \mathbf{Z}^{m \times n}$ where n is regarded as variable, but m as fixed.
("Maps variable dimension to fixed dimension" [Onn, Rothblum '05])

The landscape of computational complexity

Objective function	Variables: Dimension two	Variables: Fixed dimension	Variables: Parametric
Convex max	poly-time	poly-time	poly-time NP-hard
Convex min	poly-time	poly-time	poly-time NP-hard
Polynomial	poly-time NP-hard	FPTAS NP-hard	? ?

Concave minimization or convex maximization

Observation

Let P be a rational polytope in $\mathbf{R}^n$, and let f be such that for every $\bar{z} \in P \cap \mathbf{Z}^n$, the set $\{z \in P \mid f(z) \geq f(\bar{z})\}$ is convex. For fixed n, $\min\{f(x) \mid x \in P \cap \mathbf{Z}^n\}$ can be solved in polynomial time.

Proof

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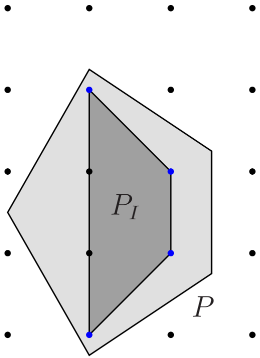
Proof

- We can find the set V of the vertices of P_I (Cook, Hartman, Kannan, McDiarmid, 1992)



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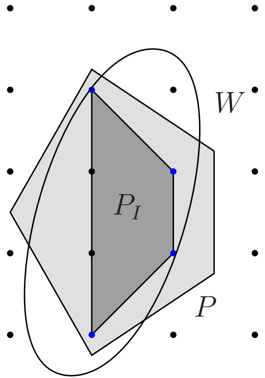
Proof

- We can find the set V of the vertices of P_I (Cook, Hartman, Kannan, McDiarmid, 1992)
- Let $\bar{z}$ be the best vertex, and let $W := \{z \in P \mid f(z) \geq f(\bar{z})\}$
- $V \subseteq W$
- As W is convex, $P_I = \operatorname{conv} V \subseteq W$



State of the art for convex minimization: general dimension

[Westerlund, Pettersson 95] [Duran, Grossmann 86, Viswanathan, G. '90, Fletcher, Leyffer '94, Bonami et al. '08]

(GP) $\min c^T z \text{ s.t. } z \in K, z \in Z = \left\{(x,y) \mid x \in X \cap \mathbf{Z}^n,\ y \in Y \subseteq \mathbf{R}^d\right\}.$

X, Y polytopes, $K = \{z \mid g_i(z) \leq 0,\ \forall\, i\}$, convex g_i (first order oracle).

Source of inspiration: [Kelly '60]

Generate sequences of points
$z^1 = (x^1, y^1), \ldots, z^l = (x^l, y^l) \in Z$
from mixed integer relaxations:

$$z^j = \arg\min\ c^T z \text{ s.t. } z \in Z,$$
$$\nabla g_i(z^k)^T (z - z^k) \leq 0,\ k < j$$

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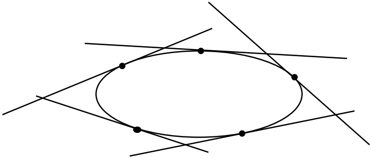

Modifications in (OA)

$z^j = (x^j, y^*)$. Replace y^* with y^j :

$$y^j = \arg \min_{z \in K \cap Z, x = x^j} c^T z \text{ or}$$

$$y^j = \arg\min_{y \in Y} \left(\max_i \{g_i(x^j, y)\} \right)$$

State of the art for convex minimization: fixed dimension

Theorem [Lenstra '83] [Grötschel, Lovász, Schrijver '88]

For any fixed $n \geq 1$, there exists an oracle-polynomial algorithm that, for any convex set $K \subseteq \mathbf{R}^n$ with $B(*, r) \subseteq K \subseteq B(0, R)$ given by a weak separation oracle, and for any rational $\varepsilon > 0$, either finds a point in $(K + B(0, \varepsilon)) \cap \mathbf{Z}^n$, or concludes that $K \cap \mathbf{Z}^n = \emptyset$.

Theorem [Khachiyan, Porkolab '00] and improvements by [Heinz '05], [Hildebrand, Köppe '12] [Dadush '13]

Let $g_1, \ldots, g_m \in \mathbf{Z}[x_1, \ldots, x_n]$ be quasi-convex polynomials of degree at most δ whose coefficients have a binary encoding length of at most s. There exists an algorithm for testing feasibility of

$$g_1(x) \leq 0, \ldots, g_m(x) \leq 0, \; x \in \mathbf{Z}^n.$$

whose running time is polynomial in m, s, δ provided that n is constant.

A general scheme for mixed integer convex minimization [Baes, Oertel, Wagner, W.] [Yudin, Nemirovskii 79]

An augmentation oracle

For a mixed integer set $\mathcal{F}$ and $x \in \mathbf{R}^n$ either **(a)** return a point $\hat{x} \in \mathcal{F}$ such that $f(\hat{x}) \leq (1+\alpha)f(x) + \delta$ or **(b)** assert non-existence.

Gradient Descent method (GDM) ($N \in \mathbf{Z}_+$, $x_0 = \hat{x}_0 \in \mathcal{F}$)

For $k = 0, \ldots, N-1$, perform the following steps:

- **Determine** $x_{k+1} = x_k - h_k \nabla f(x_k)$
- **If** $f(x_{k+1}) \geq f(x_k)$ set $x_{k+1} = x_k$, $\hat{x_{k+1}} = \hat{x}_k$ and continue.
- **If** $f(x_{k+1}) < f(x_k)$ query the oracle with input x_k.
  - **If** the oracle output is **(a)**, then update $\hat{x_{k+1}}$.
  - **If** the oracle output is **(b)**, then start gap closing :
    For $l \leq f^* \leq u$ and precision $\epsilon > 0$, find $x \in \mathcal{F}$ such that

$$f(x) - f^* \leq \epsilon.$$

Analysis and extensions [Baes, Oertel, Wagner, W.]

Theorem. For $\alpha = \delta = 0$ and f convex with Lipschitz-constant L:

If (GDM) does not terminate before N steps, then

$$f(x_{best}) - f^* \leq L\sqrt{\frac{\delta_{\mathcal{F}}}{2}} \quad \frac{ln(N) + 2}{2\sqrt{N+2} - 2}.$$

The gap-closing algorithm can be implemented to run in oracle polynomial time in $ln(\epsilon)$ and in $ln(f(x_{best}) - f^*)$.

Extensions

- We can generalize GDM to Mirror-Descent Methods, for better convergence properties.
- Constrained problems: we need a projector and a separator from the continuous feasible set.
- We allow for $\alpha, \delta > 0$, without accumulation of errors during the iterations (smallest affordable gap: $(2 + \alpha)(\alpha\hat{f}^* + \delta)$).

The continuous case without constraints

Theorem. Let f be convex and continously differentiable on its domain. Let $x^* \in$ dom f. Then, x^* attains the value

$$\min\{f(x) \mid x \in \text{ dom } f\}$$

if and only if $\nabla f(x^*) = 0$.

... and with constraints

Using KKT we are allowed to use also constraints from $\mathcal{F}$.

Implementation of the oracle: optimality condition

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The constrained mixed integer case: [Baes, Oertel, W.]

Theorem. For convex and continuously differentiable f consider

$$\min\{f(x) \mid x \in \mathcal{F}\},$$

with $\mathcal{F} = P \cap \mathbf{Z}^d \times \mathbf{R}^n$. Let $\hat{x}$ be the continuous optimum and $x^0 \in \mathcal{F}$. Then, x^0 is optimal if and only if there exist $\{x^1, \dots, x^t\} \subseteq P$ such that

- $t \leq 2^d - 1$ and $\hat{x} \in$ int L,
- the set int L is mixed-integer free,
- $f(x^i) \geq f(x^0)$ for $i \geq 1$.

$L = \{x \in P \mid \nabla f(x^i)^T(x - x^i) \leq 0\}$.

Implementation of the oracle II: [Oertel, Wagner, W.]

"MICO by MILPing"

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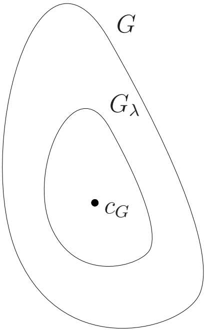
- Replace the ellispoid type method by a polytope shrinking algorithm.

The ingredients:

- For convex compact G, the centroid $c_G = \frac{\int_G x dx}{vol(G)}$.
- $G_\lambda := \lambda(G - c_G) + c_G$.



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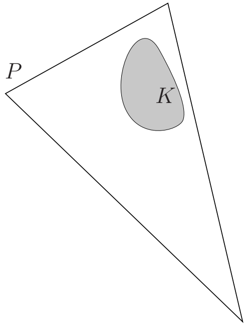

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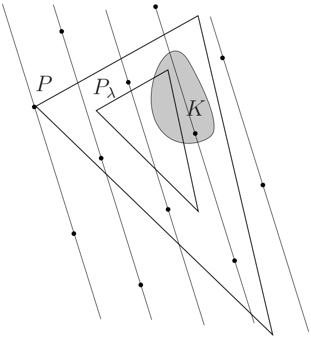

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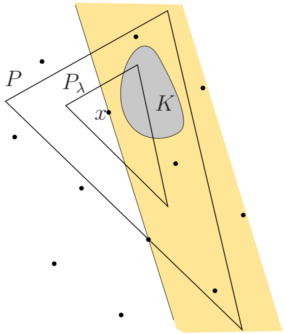
  If $x \notin K$, separate x.



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Extension of a theorem of Grünbaum 1960 ($\lambda = 0$)

Let G be a compact convex set, let H be a halfspace and let $0 < \lambda < 1$. If $G_\lambda \cap H \neq \emptyset$, then

$$\frac{\text{vol}\,(G \cap H)}{\text{vol}\,(G)} \geq (1-\lambda)^n \left(\frac{n}{n+1}\right)^n.$$

Analysis of the polytope-shrinking algorithm:

Iterations k until $\text{vol}(P) \leq \frac{1}{n!}$:

$$k \leq \frac{n\big[log(2B) + log(n)\big]}{(1-\lambda)^n(\frac{n}{n+1})^n}.$$

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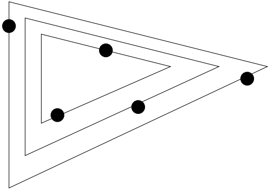

Good news about the computation of $x \in P_\lambda \cap \mathbf{Z}^n$:

- For n fixed, P_λ can be efficiently computed by solving a mixed integer linear program in dimension $n+1$:

$$\begin{aligned} t^* &= \max\ t \\ & a_i^T x + \omega(\mathbf{P}, \mathbf{a_i})t \leq b_i\ \forall i \\ & x \in \mathbf{Z}^n,\ t \geq 0. \end{aligned}$$

- (x^*, t) feasible implies (a) $x^* \in P_{1-t}$ and
- (x^*, t) feasible implies (b) $x^* \in \{x \mid x + t(P-P) \subseteq P\}$.

Polynomials in few integer variables

Integer Polynomial Programming (IPP)

$$\min\{f^k(x) \text{ subject to } x \in P \cap \mathbf{Z}^n\},$$

where f^k is a polynomial function of degree k, with integer coefficients and P is a polytope in $\mathbf{R}^n$ given by an outer description.

About the encoding of (IPP)

polyhedron P	polynomial $f^k(x)$
inequality description in binary encoding.	$f^k(x) = \sum_{i=1}^k \sum_{z\in \mathbf{Z}^n_+, \|z\|_1=i} a_z x^z$ $\forall 1 \le i \le k$ and $\forall z \in \mathbf{Z}^n_+, \|z\|_1 = i$ the input is the integer a_z in binary encoding.

Polynomiality results in dimension 2

- Theorem [Del Pia, W. '13]. (IPP) can be solved in polynomial time if $k = 2$.
- Theorem [Hildebrand, Del Pia, W., Zemmer '14] (IPP) can be solved in polynomial time if $k = 3$.
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	n=1	n=2
k=1	P	P
k=2	P	P
k=3	P	P
k=4	P	NPH
k general	P	NPH

Polynomiality results in two integer variables

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Exclusion operator

Let C be convex and P a polyhedron. In polynomial time in the encoding of P one can determine whether or not $P \setminus C \cap \mathbf{Z}^n$ is empty.

From dimension two to fixed dimension

Problem type

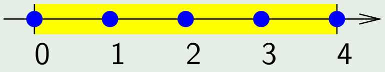
$$\begin{array}{ll} \max & f(x_1, \ldots, x_n) \\ \text{s.t.} & (x_1, \ldots, x_n) \in P \cap \mathbf{Z}^n, \end{array}$$

where

- P is a polytope,
- f is a polynomial function non-negative over $P \cap \mathbf{Z}^n$,
- the dimension n is fixed.

Generating functions

$$g_P(z_1, \ldots, z_n) = \sum_{\alpha \in P \cap \mathbf{Z}^n} z^\alpha$$



$$g_P(z) = z^0 + z^1 + z^2 + z^3 + z^4$$

Theorem (De Loera, Hemmecke, Koeppe, W. 2006)

Let n be fixed. There exists FPTAS for this problem.

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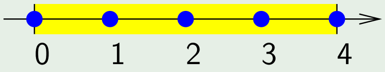

$$g_P(z) = z^0 + z^1 + z^2 + z^3 + z^4$$
$$= \frac{1 - z^5}{1 - z} \text{ for } z \neq 1$$

Theorem (De Loera, Hemmecke, Koeppe, W. 2006)

Let n be fixed. There exists FPTAS for this problem.

W-mappings

The setting: $\min \{ f(Wx) \ : \ Ax \leq b,\ x \in \mathbf{Z}^n \}$

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- Matrices $A \in \mathbf{Z}^{m \times n}$ and $W \in \mathbf{Z}^{d \times n}$, a vector $b \in \mathbf{Z}^m$

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  (Query: $\quad y^* \leftarrow \arg\min\{f(y) \ : \ By \leq c, \, y \in \Lambda\}$)

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Why and what?

- Why do we need these oracles?
- Under which conditions on the input is this problem tractable?

Assumptions about $\min f(Wx)$ subject to $x \in \mathcal{F}$.

W is in unary representation.

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The tractability question:

Conditions on $\mathcal{F}$ and A, b, resp. ?

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Theorem [Adjiashvili, Oertel, W. '14]

There is an algorithm that solves the non-linear optimization problem

$$\min \{f(Wx) \,:\, Ax \leq b,\ x \in \mathbf{Z}^n\}.$$

The number of calls of the optimization and fiber oracles is polynomial in n and $\boldsymbol{\Delta}$.

Special properties on $\mathcal{F}$ make the problem tractable.

A first polynomial time algorithm.

Let $\mathcal{F} = \left\{x \in \{0,1\}^n \mid a^T x \leq a_0\right\}$ be a knapsack set and $W \in \mathbf{Z}^{d \times n}$ encoded in unary with d fixed.

- The dual problem: $\gamma(w_0) := \min\{a^T x$ subject to $Wx = w_0\}$.
- Dynamic programming / shortest path techniques apply to the dual.
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Theorem (Lee, Onn, W. '07)

For every fixed m and p, there is an algorithm that, given $a_1, \ldots, a_p \in \mathbf{Z}$, $W \in \{a_1, \ldots, a_p\}^{m \times n}$, and a function $f : \mathbf{R}^n \to \mathbf{R}$, finds a matroid base B minimizing $f(W\chi^B)$ in time polynomial in n and $\langle a_1, \ldots, a_p \rangle$.

(... can be solved using iterated matroid intersection algorithms.)

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Integer convex minimization

- In the mixed integer setting: for $x \in \mathbf{Z}^n$, the precision used to compute $y^*(x)$ should depend on x: adaptive precision scheme.
- A mixed integer gradient method: $x \mapsto x + \lambda_k \nabla f(x)$?
- For which classes of convex functions is there an oracle polynomial algorithm if equipped with a general CO- and MILP- oracle?